THESIS FOR THE PH.D. DEGREE.

MATHEMATICAL STATISTICS.

" Kapteyn's Theory of Skew Frequency ,and

Orthogonal Polynomials in One and Two

Variables."

Submitted to Edinburgh University,

1940.

by L.R. Shenton, B.Sc.

"Kapteyn's Theory of Skew Frequency, and Orthogonal Polynomials in One and Two Variables."

Chapter 1. SKEW FREQUENCY pages 1-61

Chapter 2. PROPERTIES OF ORTHOGONAL POLYNOMIALS RELATED TO THEIR GENERATING FUNCTIONS. pages 62-97

Chapter 3. ORTHOGONAL POLYNOMIALS IN TWO VARIABLES. pages 98- 144

Summary . pages 145-148

and

References to Literature.

CHAPTER 1. INTRODUCTORY.

KAPTEYN'S THEORY OF SKEW FREQUENCY.

The representation of frequency distributions has received the attention of many writers. Chief among them we may mention Pearson who approached the problem by considering the type of differential equation which they satisfy. He considered

$\frac{dy}{dx}\frac{1}{y} = \frac{x-a'}{ax^2+bx+c}$ where $y=f(x)$ represents the distribution.

For different values of a',a,b,c the Peassonian system of skew curves arises, and by considering the discriminant of the quadratic it can be decided which particular type will lead to the best approximation to the set of data.

In "Skew Frequency Curves in Biology and Statistics" by J.C.Kapteyn Sc.D. ,1903 ,the idea is put forward that skewness in frequency curves is due to some transformation of the variable under consideration. As a simple example , suppose that a large number of berries of a fruit are such that the radii are normally distributed. If y is the frequency ordinate corresponding to a given radius r, then since the distribution is normal

$$y= ce^{-ar^2} \quad \text{where } c \text{ and } a \text{ are greater than } 0.$$

The frequency between r and $r+dr$ is

$$ce^{-ar^2}dr$$

How will the surface areas and volumes be distributed?

$S=$ surface area$=Ar^2$ and $V=Br^3$ where A and B are constants and we have assumed that the berries are approx. spherical.

The 'surface areas' distribution will be

$$dy = \frac{ce^{-bS}\, dS}{\sqrt{S}} \qquad \text{with skewness}$$

$$b_1 = \frac{u_3^2}{u_2^3} = \int_{-\infty}^{\infty} cA^3 r^6 e^{-ar^2}\, dr \div \int_{-\infty}^{\infty} cA^2 r^4 e^{-ar^2}\, dr$$

$$= \frac{\Pi}{c}\;\frac{25}{4}\sqrt{\frac{a}{\Pi}}$$

It is clear that the surface areas will exhibit positive skewness and similarly this is also a property of the volume distribution.

We proceed to a more general treatment of Kapteyn's approach.

2. A GENERAL KAPTEYN TRANSFORMATION.

The ordinary normal theory arises from the hyphothesis of a linear additive variate compound by a great number of random increments which are constant. In the binomial case; the generating function is

$$G(t) = (p_1 t^{\Delta x} + q_1)(p_2 t^{\Delta x} + q_2)(p_3 t^{\Delta x} + q_3) \ldots\ldots\ldots\ldots (p_n t^{\Delta x} + q_n)$$

where Δx is a constant, and the sum of x such increments is x. When this x is expressed as a standardized deviate from its mean value $\sum_{i=1}^{n} p_i$ the coefficient of t^x in G(t) tends to

$$p = \frac{1}{\sqrt{2\Pi}}\, e^{-x^2/2} \quad \text{the normal distribution}$$

or to $\quad p = \frac{e^{-m} m^x}{x!} \quad$ the Poisson distribution.

When the increments are not constant but variable, then we can compound probabilities and add increments as before in

$$G(t) = (p_1 t^{\Delta f_x} + q_1)(p_2 t^{\Delta f_x} + q_2)(p_3 t^{\Delta f_x} + q_3) \ldots\ldots\ldots\ldots (p_n t^{\Delta f_x} + q_n)$$

where it is understood that x increments Δf_x must now be regarded as summing not to x but to f(x). Assuming a monotonic transformation, a parallelism can be made between the case Δf_x= constant and the present case, whence it is seem that the probability differential of the former x increments Δx is that of the x increments Δf_x. Hence to obtain the probabili -ty differential we must carry out a functional transformation on the probability differential.

We have $G(t) = S\left[p(x)\, t^{f(x)}\right]$ in the discrete case,

$= \int p(x) t^{f(x)}\, dx$ in the continous case.

which is the generating function of Kapteyn function

$$p(\ g(X)\ )\ \frac{dx}{dX} \qquad \text{where } x=g(X) \text{ and } X= f(x)$$

Kapteyn gave special attention to the case

$g(X) = (X+k)^q$ with frequency function

$$y = \frac{h}{\sqrt{\Pi}}\, q\, (x+k)^{q-1}\, e^{-h^2(\ (x+k)^q\ -\ M\ )^2} \qquad (A)$$

which leads to five special cases as follows-

(1) q positive. (2) q negative. (3) q = 0

(4) q= infinity (5) q= -infinity.

for which we have the following corresponding g(x) 's

(1) $(x+k)^q$ (2) $-(x+k)^q$ (3) $\log(x+k)$

(4) $e^{a(x-b)}$ (5) $-e^{a(x-b)}$

To determine the constants h, M, k and q in the expression (A) above, Kapteyn equates the values of four relative frequencies for the data and the transformed curve. For example , if $F_{x_1}^{x_2}$ represents the frequency of x for

$x_1 \quad x \quad x_2$ then

$$A\int_{-k}^{x} (x+k)^{q-1} e^{-h^2\left((x+k)^q - M\right)^2} dx = F_{-k}^{x}$$

and putting $(x+k)^q - M = zh^{-1}$ then

$$F_{-k}^{x} = A\int_{-hM}^{h\left((x+k)^q - M\right)} e^{-z^2} dz$$

Taking for x four values $x_{1,}\ x_2,\ x_3,\ x_4$ we are led to

equations to determine h, k, q and M thus

$$h\left((x_1+k)^q - M\right) = R_1 \qquad h\left((x_3+k)^q - M\right) = R_3$$
$$h\left((x_2+k)^q - M\right) = R_2 \qquad h\left((x_4+k)^q - M\right) = R_4$$

which except for the case q=0 can only be solved graphically. If in advance we assume q=0 then we can fit a curve by the method of moments as has been shewn by S.D. Wicksell . (See Ref.4 at the end.) Wicksell has given a condition which must be satisfied by the skewness and excess of the data for this transformation to be successful.

3. THE FORM OF THE TRANSFORMATION ASSUMED.

We consider a series form for the transformation , namely

$$g(x) = a + bx + cx^2 + dx^3 + \ldots\ldots\ldots\ldots\ldots$$

with frequency function

$$y = A\,\frac{dg(x)}{dx}\,e^{-B\,(g(x))^2}$$ and in special cases of

distributions with one abrupt tail we consider the case

$$g(x) = h^{-1}\log(\,a + bx + cx^2 + dx^3 + \ldots\ldots)$$

Generally speaking the first transformation will suffice to give a good approximation to the data, but cases of failure arise for which the logarithmic transformation is successful. Neither of the transformations can be expected to graduate a "J" shaped distribution as will be made clear later.

Now

$$(\text{ Frequency })_{x_1}^{x_2} = A\int_{g(x_1)}^{g(x_2)} e^{-t^2}\,dt$$ where A is independent of x. We determine the constants of g(x) by a systematic method set out by Dr. A.C.Aitken in (On the Graduation of Data by the Orthogonal Polynomials of Least Squares. Ref. 2)

Suppose that the data is

x =	x_1	x_2	x_3	x_4	x_n
Freq.	f_1	f_2	f_3	f_4	f_n

where $x_i = i$ for i = 1,2,3, n.

Then $$\frac{\displaystyle\mathop{S}_{i=1}^{i=n} f_i}{\displaystyle\mathop{S}_{i=1}^{i=m} f_i} = \frac{1}{\sqrt{\Pi}}\int_{g(x_m)}^{g(x_n)} e^{-t^2}\,dt = \frac{1}{\Pi}\int_{-\infty}^{R_n} e^{-t^2}\,dt$$

$= f_i$

After reference to tables of $\dfrac{1}{\sqrt{\Pi}}\displaystyle\int_{-\infty}^{\lambda} e^{-t^2}\,dt$

as found in Kapteyn's paper , we shall have n values of

$R_i = g(x_i) = a + bx_i + cx_i^2 + \dots\dots\dots\dots\dots\dots$

The problem of finding the best values of a,b,c, etc.,

is resolved by Least Squares and Orthogonal Polynomials as

set out in the paper by Dr. Aitken referred to. Briefly if

$g(x) = a_o + a_1\ T_1(x) + a_2T_2(x) + a_3T_3(x) + \dots\dots$

where $T_r(x)$ is the rth. Tchebychef polynomial for a set of

n data, then

$$a_r = \frac{S\ R\ T_r(x)}{S\ \ T^2\ (x)}$$ where S represents summation over x

and $R = g(x_r)$ and is the solution of

$$1/\sqrt{\pi}\int_{-\infty}^{R} e^{-t^2}\ dt\ = f_i'$$

and $$\sum_{x=o}^{x=n-1} \left(R - (a_o + a_1T_1(x) + a_2T_2(X) + \dots\dots) \right)^2 = \text{Residual}$$

is a minimum.

Taking into the account the orthogonal relations

we have $\quad$ Residual $= S\ R^2 - a_oS\ R\ - a_1S\ RT_1(x)\ -a_2S\ RT_2(x)..$

and so the residual at the determination of each a_i can

be examined and we are satisfied with the a_i which gives

the least value for $\quad \dfrac{\text{Residual}}{n-i-1}$.

The convergence or otherwise of the transformation

$$g(x) = a_o + a_1T_1(x) + a_2T_2(x) + \dots\dots\dots\dots$$

need not be examined since we envisage only a finite number

of terms; in general there can be little point in proceeding

beyond a quintic transformation for further terms would involve the sixth, seventh ..... moments of the R data which would be subject to a considerable degree of error. We now turn to several practical examples set out in full detail.

4 A. PRACTICAL EXAMPLES OF KAPTEYN TRANSFORMATIONS.

Example 1. Data: "The number of entrants, limited payment policies, 1863-93." (Elderton, Frequency Curves, Third Ed. P.77)

No. of Entrants ÷ 100.	$(\text{Freq.})_{-\infty}^{x_i}$	'D'	R
1	1	:0027	-2:7800
56	57	:1549	-1:0157
167	224	:6087	0:2759
98	322	:8750	1:1505
34	356	:9674	1:8443
9	365	:9918	2:4000
2	367	:9973	2:7800
1	368		
S=368			

$$\text{where } 1/\sqrt{2\pi}\int_{-\infty}^{R} e^{-x^2/2}\,dx = \frac{(\text{Freq.})_{-\infty}^{x_i}}{(\text{Freq.})_{-\infty}^{\infty}} = D \text{ (say)}$$

(N.B. Throughout these examples we shall use the sign : for the usual single decimal point.)

Having determined R from tables of the normal integral we proceed to find the moments of R and thus a_o, a_i, etc.

R	$_u\hat{\Sigma}_1$	$\hat{\Sigma}_1^2$	$_u\hat{\Sigma}_1^3$	$\hat{\Sigma}_1^4$	R^2
-2:7800	-2:7800	-2:7800	-2:7800	-2:7800	7:7284
-1:0157	-3:7957	-6:5757	-9:3557	-12:1357	1:0316
0:2759	-3:5198	-10:0955	(-14 :40345)		0:0761
1:1505	(-2:94455)				1:3237
1:8443	(7:59955)				3:4014
2:4000	7:0243	14:9843	(18:23215)		5:7600
2:7800	5:1800	7:9600	10:7400	13:5200	7:7284
	2:7800	2:7800	2:7800	2:7800	
					S=27:0496

Adding and subtracting corresponding entries we have

$m_{\{0\}} = 4:6550$ $m_{\{1\}} = 25:0798$ $m_{\{2\}} = 3:8287$ $m_{\{3\}} = 25:6557.$

We have used Central Factorial Moments so we set out the table of Centred Tchebychef Polynomials for n=7.

Table of Central Values of Tchebychef Polynomials, n=7

a→ ↓m(x)	0:6650	0;44785	-0:04350	0:004799
4:6550	1		-12	
25:0798		2		-20
3:8287			6	
25:6557				20
$S\ T_r^2(x)$	7	112	756	2400

We have also-

V_o=1:1870 $\mu\&W_o$=0:79972 $\&^2V_o$=-0:2610 $\mu\&^3W_o$= 0:09598

Residual Table.

	$a_iS\ R\ T_i(x)$	R^2	Residual n-i-1
a_o = 4:6550/7 =0:6650	3:0956	23:9540	
$a_i = \frac{2\text{x}25:0798}{112}$ =0:44785	22:4640	1:4900	0:298
$a_2 = -\frac{32:8878}{756}$ =-0:04350	1:4306	0:0594	0:0148
$a_3 = \frac{0:5759 \text{ x } 20}{2400}$ = 0:004799	0:0553	0:0041	0:oo13

(N.B. We could proceed further and find a_4 but with a short set of data as in this case the increase in accuracy would scarcely be substantial since the residual after a_3=:0041)

The two half-tables for V and W are as follows-

V	&V	$\&^2$V	W	&W	$\&^2$W	$\&^3$W
1:1870		-0:2610	0		0	
	-0:1305			:79972		:09598
1:0565		-0:2610	0:79972		:09598	
	-0:3915			:89570		:09598
0:6650		-0:2610	1:69542		:19196	
	-0:6525			1:08766		
0:0125			2:78308			

Adding and subtracting correspoinding values of V and W we find the computed values of R thus -

R computed	$1/\sqrt{2\pi}\int_{-\infty}^{R} e^{-x^2/2}dx$	$(R_{comp.} - R_{actual})^2$	Frequency.	Observed
-2:7706	:0028	:000,088	1	1
- 1:0304	:1514	:000,216	55	56
0:2568	:6014	:000,365	166	167
1:1870	:8824	:001,332	103	98
1:8562	:9683	:000,142	32	34
2:3604	:9909	:001,568	8	9
2:7956	:9974	:000,243	2	2
	(:9993)	S= :003,954	1	1
			S=368	

We observe that S (R computed - R actual $)^2$ =:003,954 is in good agreement with the Residual =:0041 , which provides a check on the calculations.

Equation of the Curve.

The equation of the curve which represents the distribution is

$$y = \frac{1}{\sqrt{2\pi}} R'(x)\, e^{-\frac{1}{2}R^2(x)}$$

where $R(x) = 0{:}665 + 0{:}44785T_1(x) - 0{:}0435T_2(x) + 0{:}004799T_3(x)$

$$= 1{:}1870 + 0{:}78372x - 0{:}1305x^2 + 0{:}015997x^3$$

and $R'(x) = \frac{dR}{dx} = 0{:}78372 - 0{:}2610x + 0{:}04799x^2$

Hence

$$y = \frac{(:78372 - :2610x + :04799x^2)}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(1:1870 + :78372x - :1305x^2 + :0159\,9x^3)^2}$$

To draw a graph of the curve we look up values of $1/\sqrt{2\pi}$ Exp. $-\frac{1}{2}r^2$ and multiply by the corresponding value of $:78372 - :2610x + :04799x^2$. The table for this calculation if given overleaf.

Calculation of Ordinates of the Curve.

x	:78372	$:04799x^2$	$-:261x$	R'(x)	$e^{-\frac{1}{2}R^2(x)}/\sqrt{2\pi}$	y
-3	:78372	:43191	:783	1:99863	:0086	:0172
-2	:78372	:19196	:522	1:49768	:2346	:3514
-1	:78372	:04799	:261	1:09271	:3860	:4218
0	:78372	00000	000	0:78372	:1972	:1545
1	:78372	:04799	-:261	0:57071	:0712	:0406
2	:78372	:19196	-:522	0:45368	:0246	:0112
3	:78372	:43191	-:783	0:43263	:0080	:0035

In addition we give the value of $y(-3/2) = R'(-3/2)\ \frac{\text{Exp.}}{\sqrt{2\pi}} -\frac{1}{2}R^2(-3/2)$

$= 1{:}2832 \times 0{:}3770 = 0{:}4838.$

(N.B. It should be noted that the curves which result from Kapteyn transformations are not difficult to graph since the only calculations to be made are those of R'(x) . On the other hand the maximum ordinate occurs at that value of x which satisfies

$$R''(x) = R'(x)^2\ R(x)$$

which is an equation in x of degree 3p-2 where p is the degree of the transformation R(x). For example in the case of a cubic transformation ,the above equation for the maximum ordinate's x value is of degree seven and a ready solution would not in general be possible.)

It is of interest to compare the Kapteyn graduation with that of Type vl of the Pearsonian system as computed in Elderton.

Observed.	Kapteyn.	Pearson's Type 6.
1	1	1
56	55	50
167	166	168
98	103	100
34	32	36
9	8	10
2	2	2
1	1	$\frac{1}{2}$

Test of Goodness of Fit.

$$\text{Computing } \chi^2 = S \frac{(\text{Freq. computed} - \text{Freq. observed})^2}{\text{Freq. Computed}}$$

we find $\chi^2 = 0{:}4826$

and $n = 5-4 = 1$ = number of degrees of freedom

$$P_{\chi^2} = 0{:}49$$

We may conclude that the graduation is a satisfactory explanation of the facts.

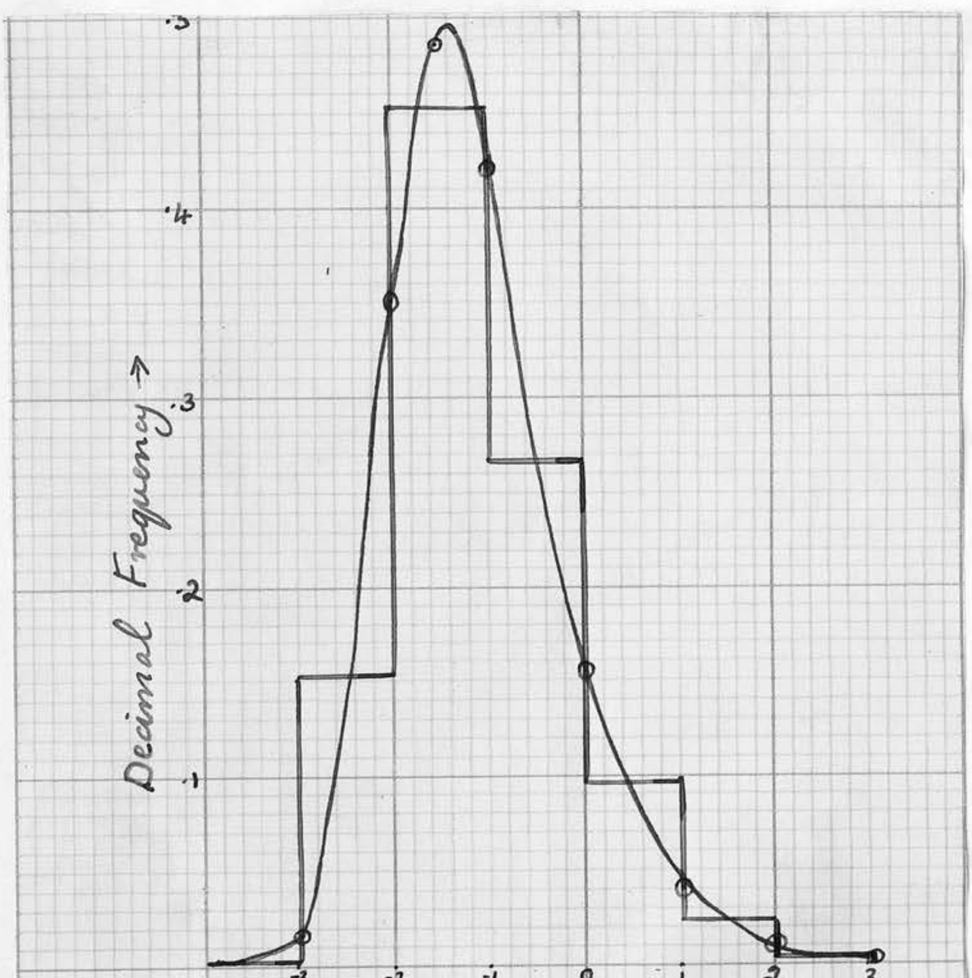
We give the graph of the curve and the distribution on the next page.

GRAPH OF THE CURVE AND DISTRIBUTION.

Data: "The number of entrants, limited payment policies, 1863-93."

Kapteyn curve is :-

$$y = \frac{1}{\sqrt{2\Pi}}(:78372-:2610X+:04799X^2)e^{-\frac{1}{2}(1:1870+:78372X-:1305X^2+:015997X^3)^2}$$



Example 2. Data:- "Game of Patience" 500 tries. H.E.H Greenleaf, Annals of Maths. 1932.

Frequency.	First Sums.	Decimal	R
3	3	:006	-1:7766
7	10	:02	-1:4528
35	45	:09	-0:9483
101	146	:292	-0:3871
89	235	:470	-0:0532
94	329	:658	0:2879
70	399	:798	0:5900
46	445	:890	0:8673
30	475	:950	1:1633
15	490	:980	1:4528
4	494	:988	1:5960
5	499	:998	2:0400
1	500		

In this example we use tables of $1/\sqrt{\pi}\int_{-\infty}^{R} e^{-x^2}\,dx$

as found in Kapteyn's paper. We proceed to find the moments of R using central factorial methods.

R	$\hat{\Sigma}_1$	$u\hat{\Sigma}_1^2$	$\hat{\Sigma}_1^3$	$u\hat{\Sigma}_1^4$	$\hat{\Sigma}_1^5$	$u\hat{\Sigma}_1^6$
-1:7766	-1:7766	-1:7766	-1:7766	-1:7766	-1:7766	-1:7766
-1:4528	-3:2294	-5:0060	-6:7826	-8:5592	-10:3358	-12:1124
-0:9483	-4:1777	-9:1837	-15:9663	-24:5255	-34:8613	-46:9737
-0:3871	-4:5648	-13:7485	-29:7148	-54:2403	-89:1016	(-91:5245)
-0:0532	-4:6180	-18:3665	-48:0813	(-78:2809)		
0:2879	-4:3301	(-20:5315)				
0:5900	7:7094	(27:9910)				
0:8673	7:1194	24:1363	59:6340	(93:5515)		
1:1633	6:2521	17:0169	35:4977	63:7345	103:7673	(105:7524)
1:4528	5:0888	10:7648	18:4808	28:2368	40:0328	53:8688
1:5960	3:6360	5:6760	7:7160	9:7560	11:7960	13:8360
2:0400	2:0400	2:0400	2:0400	2:0400	2:0400	2:0400

From these we derive:-

$m_{\{0\}}$ =3:3793 $m_{\{1\}}'$ =48:5225 $m_{\{2\}}$ = 11:5527 $m_{\{3\}}'$ =171:8324

$m_{\{4\}}$ =14:6657 $m_{\{5\}}'$ = 197:2769

The value of $S(R^2)$ is also required .

R^2
3:15631
2:11063
0:89927
0:14985
0:00283
0:08289
0:34810
0:75221
1:35327
2:11063
2:54722
4:16160
17:67481

Table of Central Values and Differences ,Tchebychef polynomials.

n=12

$\frac{a_r}{m}$ —	0:28161	0:16966	-0:00408	0:00031	0:00016	-0:00003
3:3793	1		-35		280	
				-70		
48:5225		2				560
11:5527			6		-160	
171:8324				20		-448
14:6657					70	
197:2769						252
$S\ T_r^2(x)$	12	572	12012	128700	800800	3118752

Combining the values of the a's in this table by multiples of rows we find , using the quartic transformation ,

$\mu V_o = 0:46921$ $\quad \delta W_o = 0:31762$ $\quad \mu\delta^2 V_o = -0:05008$

$\delta^3 W_o = 0:00620$ $\quad \mu\delta^4 V_o = 0:01120$

The residual table is now given from which it is seem that the quartic fit is likely to lead to the best results.

		$a_iS\ uT_i(x)$	Residual	Residual / n-i-1
$a_0 = \frac{3:3793}{12}$	=0:28161	0:95164	16:72317	
$a_1 = \frac{97:0450}{572}$	=0:16966	16:46465	0:25852	0:0258
$a_2 = -\frac{48:9593}{12012}$	=-0:00408	0:19975	0:05877	0:00653
$a_3 = \frac{40:0730}{128700}$	=0:00031	0:01242	0:04635	0:00579
$a_4 = \frac{124:3710}{800800}$	=0:00016	0:01990	0:02645	0:00378
$a_5 = -\frac{94:5364}{3118752}$	=-0:00003	0:00284	0:02361	0:00395

By examination of $\frac{\text{Residual}}{n-i-1}$ it is clear that the transformation to choose is the one up to and including a_4. We proceed to compute R from two Half Tables of Differences for V and W.

Half Tables for V and W.

V	&v	&^{2}V	&^{3}V	&^{4}V
	0		0	
:46921		-:05008		:01120
	-:05008		:01120	
:41913		-:03888		:01120
	-:08896		:02240	
:33017		-:01648		:01120
	-:10544		:03360	
:22473		+:01712		:01120
	-:08832		:04480	
:13641		:06192		:01120
	-:02640		:05600	
:11001		:11792		
	+:09152			
:20153				

W	&W	$\&^2$W	$\&^3$W
	:31762		:0062o
:15881		:00310	
	:32072		:00620
:47953		:00930	
	:33002		:00620
:80955		:01550	
	:34552		:00620
1:15507		:02170	
	:36722		:00620
L:52229		:02790	
	:39512		:00620
1:91741		:0341	
	:42922		
2:34663			

Combining corresponding values of V and W we find R computed.

R computed.	x	Decimal	Frequency. computed	Actual	Greenleaf	Check column (Rcomp.-R act.)^2
-2:1451	-6:5	:0012	1			:00095
-1:8074	-5:5	:0053	2	3		:00448
-1:3859	-4:5	:0250	10	7	9	:00032
-0:9303	-3:5	:0941	34	35	44	:00852
-0:4794	-2:5	:2489	77	101	86	:00005
-0:0604	-1:5	:4660	109	89	99	:00051
+0:3104	-0:5	:6696	102	94	87	:00144
0:6280	+0:5	:8127	71	70	73	:00099
0:8987	1:5	:8982	43	46	52	:00056
1:1397	2:5	:9465	24	30	26	:00533
1:3798	3:5	:9745	14	15	11	:00393
1:6587	4:5	:9906	8	4	8	:00016
2;0274	5:5	:9980	4	5	4	
2:5482	6:5	:9998	1	1		S= :0272 and residual =:0265

Greenleaf computes S (Frequency -Computed Freq.)^2 and finds 562 for his curve as compared with 1400 for a fitting by Pearson.

This figure for the Kapteyn curve is 1113. It is evident that the curve fitted by Greenleaf is a better approximation to the data than either Pearson's or Kapteyn's, but it is worth noting that Greenleaf uses seven parameters as compared with five used in the Kapteyn quartic transformation. Greenleaf remarks that the value of S $(\text{Freq.}-\text{Comp.Freq.})^2$ for his fourth degree curve is 1170 which is near the Kapteyn figure 1113 and also directly comparable. Moreover there can be little doubt that fitting a curve of the type

$$y = \left(a_o G_o(x) + a_1 G_1(x) + a_2 G_2(x) + \ldots\ldots\right) \phi(x)$$

where $G_r(x)$ is the rth Gram polynomial orthogonal w.r.t. $\phi(x)$ the coefficient of t^x in $(pt + q)^n$ $p=q=\frac{1}{2}$, involves heavier calculations than does the Kapteyn graduation.

<u>EQUATION of the Kapteyn Curve.</u>

The transformation is $g(x) = R(x) =$

$$:46921 + :31762x - :02504(x^2-\tfrac{1}{4}) + :0010\dot{3}x(x^2-\tfrac{1}{4}) + :0004\dot{6}(x^2-\tfrac{1}{4})(x^2-9/4)$$

$$R'(x) = :31736 - :02620\dot{6}(2x) + :000774\dot{9}(2x)^2 + :0002\dot{3}(2x)^3$$

and

$$y = 1/\sqrt{\pi}\quad R'(x)\quad e^{-R^2(x)}$$

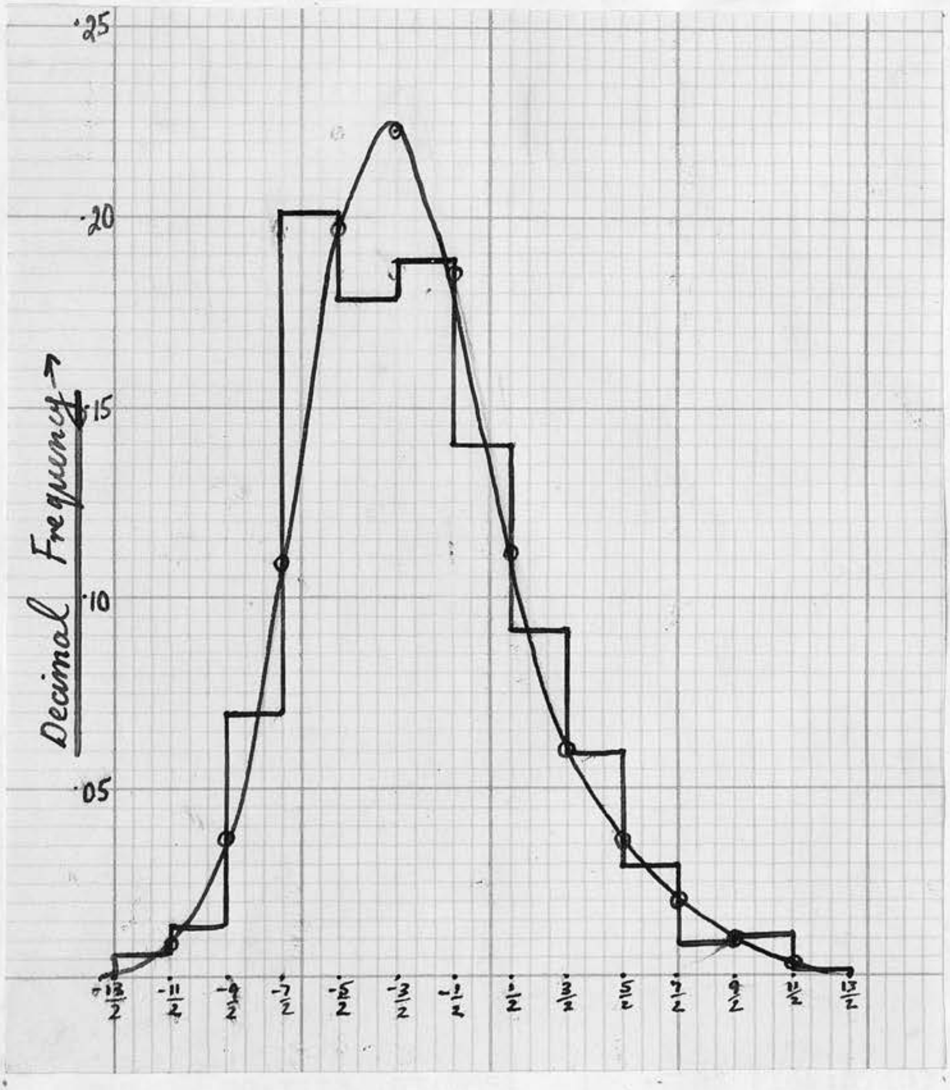
<u>Ordinates of the Curve.</u>

x=	-11/2	-9/2	-7/2	-5/2	-3/2	-1/2	+1/2	3/2
y=	:0084	:0367	:1089	:1967	:2230	:1850	:1111	:0634

x=	5/2	7/2	9/2	11/2
y=	:0362	:0212	:0113	:0040

The graph of the curve and distribution is given overleaf.

GRAPH OF THE KAPTEYN CURVE FITTED TO GREENLEAF'S "GAME OF PATIENCE" DATA



Example 3 Data:- Weights for Scotland, Yule, Statistics.

x	Frequency	First Sum	Decimal	R	R^2
-6	1	1	0008	-2:2361	5:00014
-5	8	9	0074	-1:7233	2:96976
-4	22	31	0256	-1:3790	1:90164
-3	63	94	0776	-1:0052	1:01043
-2	173	267	2203	-0:5452	0:29724
-1	255	522	4307	-0:1234	0:01523
0	275	797	6576	0:2871	0:08243
1	168	965	7962	0:5855	0:34281
2	125	1090	8993	0:9033	0:81595
3	67	1157	9546	1:1957	1:42970
4	24	1181	9744	1:3787	1:90081
5	14	1195	9860	1:5540	2:41492
6	7	1202	9917	1:6933	2:86726
7	4	1206	9950	1:8200	S=21:04832
8	2	1208	9967	1:9200	
9	4	1212			
	S=1212				
	(1)	(2)	(3)	(4)	(5)

The relation between column (3) and (4) is $1/\sqrt{\pi}\int_{-\infty}^{R} e^{-t^2}\,dt$ = Col. (3)

Column (5) gives the value of S^2 and we propose to graduate the data from x=-6 to x=+6, thus excluding the small frequencies for x= 7,8 and 9 from the graduation. We next set out the mean Central Factorial Moments of R.

R	$u\hat{\Sigma}_1$	$\hat{\Sigma}_1^2$	$u\hat{\Sigma}_1^3$	$\hat{\Sigma}_1^4$	$u\hat{\Sigma}_1^5$	$\hat{\Sigma}_1^6$
-2:2361	-2:2361	-2:2361	-2:2361	-2:2361	-2:2361	-2:2361
-1:7233	-3:9594	-6:1955	-8:4316	-10:6677	-12:9038	-15:1399
-1:3790	-5:3384	-11:5339	-19:9655	-30:6332	-43:5370	-58:6769
-1:0052	-6:3436	-17:8775	-37:8430	-68:4762	-112:0132	-170:6901
-0:5452	-6:8888	-24:7663	-62:6093	-131:0855	(-177:5559)	
-0:1234	-7:0122	-31:7785	(-78:4985)			
	(-6:8687)					
0:2871						
	(7:4540)					
0:5855	7:3105	29:4238	(68:4140)			
0:9033	6:7250	22:1133	53:7021	109:8186	(145:5740)	
1:1957	5:8217	15:3883	31:5888	56:1165	90:6647	136:9267
1:3787	4:6260	9:5666	16:2005	24:5277	34:5482	46:2620
1:5540	3:2473	4:9406	6:6339	8:3272	10:0205	11:7138
1:6933	1:6933	1:6933	1:6933	1:6933	1:6933	1:6933

Adding and subtracting corresponding entries we have

$m_{\{0\}}'$=0:5853 $m_{\{1\}}$=61:2023 $m_{\{2\}}'$=-10:0845 $m_{\{3\}}$=240:9041

$m_{\{4\}}'$=-31:9819 $m_{\{5\}}$=307:6168

Central Table for n=13

a_r	:045,023.	:168.138.	-:004,722.	-:000,341.	:000014	:000021
0:5853	1		-42		420	
61:2023		2		80		720
-10:0845			6		-200	
240:9041				20		-504
-31:9819					70	
307:6168						252
S $T_r^2(x)$	13	728	18018	1701700	228800	8019648

Residual Table.

	a_iS RT_i	S^2	$S^2/(n-i-1)$
a_0= $\frac{:5853}{13}$ =:045,023	:026,352	21:021,968	
a_1= $\frac{122:4046}{728}$ =:168,138	20:580,865	0:441,103	:0401
a_2=-$\frac{85:0896}{18018}$ =-:004,722	0:401,793	0:039,310	:0039
a_3=-$\frac{78:1020}{228800}$ =-:000,341	0:026,633	0:012,677	:0014
a_4= $\frac{23:99\ 3}{1701700}$ =:000,014,099	0:000,338	0:012,339	:0015
a_5= $\frac{169:4232}{8019648}$ =:000,021,126	0:003,579	0:008,760	:0012

From this table it is clear that the cubic transformation is better than the quartic and only slightly inferior to the quintic. We graduate with the cubic since the possible refinement gained by proceeding to the quintic is not sufficient to warrant the extra computational work.

We find V_0 =:243,347 μ&W_0=:363,556 &$^2V_0$=-:028,332

μ&$^3W_0$=-:006,820

Using these values the computed R are found from two half tables of differences for V and W which are now given.

V	&V	$\&^2$V	W	&W	$\&^2$W	$\&^3$W
:243,347		-:028,332	0		0	
	-:014,166			:363,556		-:006,820
:229,181		-:028,332	:363,556		-:006,820	
	-:042,498			:356,736		-:006,820
:186,683		-:028,332	:720,292		-:013,640	
	-:070,830			:343,096		-:006,820
:115,853		-:028,332	1:063,388		-:020,460	
	-:099,162			:322,636		-:006,820
:016,691		-:028,332	1:386,024		-:027,280	
	-:127,494			:295,356		-:006,820
-:110,803		-:028,332	1:681,380		-:034,100	
	-:155,826			:261,256		-:006,820
-:266,629		-:028,332	1:942,636		-:040,920	
	-:184,158			:220,336		-:006,820
-:450,787		-:028,332	2:162,972		-:047,740	
	-:212,490			:172,596		
-:663,277			2;335,568			

Hence we find the computed R as follows ,and thence the frequency.

R compt.	(R compt. - R act. $)^2$	Decimal	Frequency	Observed	$(F-O)^2/F$
-2;2093	:000,718	:0009	1	1	
-1:7922	:004,747	:0057	6	8	:57
-1:3693	:000,094	:0264	25	22	:36
-0:9475	:003,329	:0902	77	63	2:55
-0:5336	:000,135	:2253	164	173	:49
-0:1344	:000,121	:4246	242	255	:70
0:2433	:001,918	:6347	255	275	1:57
0:5927	:000,052	:7991	199	168	4:83
0:9070	:000,014	:9002	123	125	:03
1:1792	:000,272	:9523	63	67	:25
1:4027	:000,576	:9763	29	24	:86
1:5706	:000,276	:9868	13	14	:08
1:6760	:000,299	:9911	5	7	:27
	:012,551		$\frac{1}{9}$	$\frac{4}{6}$	12:56

and S=:012,677

It is to be observed that the Kapteyn graduation spreads the positive tail slightly ,but apart from this the agreement is good. Applying the test,we have

= 12:56 n= 12-4= 8

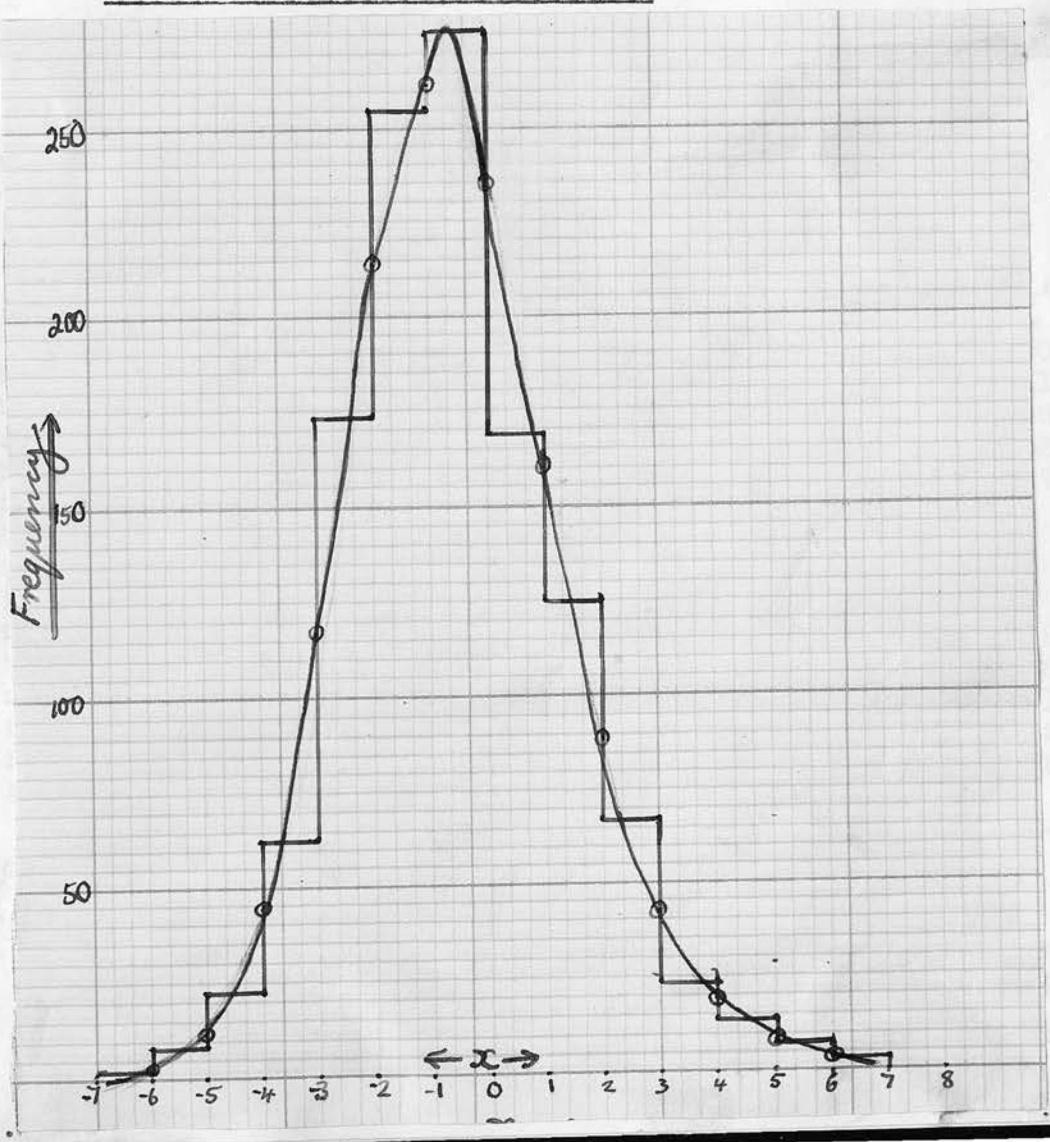
P = 0:13.

and so we can say that the fitted curve approximately accounts for the facts. The equation of the curve is

$$y = 1 / \sqrt{\pi} \frac{dR}{dx} e^{-R^2(x)}$$

where $R(x) = :243347 + :364693x - :014166x^2 - :001137x^3$

"WEIGHTS FOR SCOTLAND " Graph.



Example 4. Data- "On the rate of mortality among female nominees etc. " J. Inst. Act. xxxlll 262-8 . Elderton "Frequency Curves" page 97 .

x=	Frequency	First Sums	Decimal	R	R^2
-6:5	1	1	0005	-3:3086	10:94683
-5:5	5	6	0028	-2:7700	7:67290
-4:5	8	14	0065	-2:4850	6:17523
-3:5	12	26	0120	-2:2567	5:09270
-2:5	28	54	0250	-1:9600	3:84160
-1:5	82	136	0629	-1:5308	2:34335
-0:5	128	264	1221	-1:1645	1:35606
+0:5	253	517	2391	-0:2603	0:50325
+1:5	342	859	3973	+0:3587	0:06776
+2:5	525	1384	6401	1:0056	0:12867
+3:5	438	1822	8427	1:8157	1:01123
+4:5	265	2087	9653	2:3200	3:29677
+5:5	53	2140	9898	2:9000	5:38240
+6:5	18	2158	9981		8:41000
+7:5	4	2162			56:22875
+	2162				

Where 'Decimal' $= 1 \,/\sqrt{\Pi} \int_{-\infty}^{R} e^{-x^2/2}\,dx$

CENTRAL MOMENTS OF "R"

R	$\hat{\Sigma}_1$	$\mu\hat{\Sigma}_1^2$	$\hat{\Sigma}_1^3$	$\mu\hat{\Sigma}_1^4$	$\hat{\Sigma}_1^5$
-3:3086	-3:3086	-3:3086	-3:3086	-3:3086	-3:3086
-2:7700	-6:0786	-9:3872	-12 :6958	-16:0044	-19:3130
-2:4850	-8:5636	-17:9508	-30:6466	-46:6510	-65:9640
-2:2567	-10:8203	-28:7711	-59:4177	-106:0687	-172:0327
-1:9600	-12:7803	-41:5514	-100:9691	-207:0378	-379:0705
-1:5308	-14:3111	-55:8625	-156:8316	(-285:4536)	
-1:1645	-15:4756	(-63:6003)			
-0:7094	7:4303	(43:4518)			
-0:2603	8:1397	39:7367	120:7064	(230:7913)	
+0:3587	8:4000	31:5970	80:9697	170:4381	316:8222
1:0056	8:0413	23:1970	49:3727	89:4684	146:3841
1:8157	7:0357	15:1557	26:1757	40:0957	56:9157
2:3200	5:2200	8:1200	11:0200	13:9200	16:8200
2:9000	2:9000	2:9000	2:9000	2:9000	2:9000

$m_{\{0\}} = -8:0453$ $m_{\{1\}}' = 107:0521$ $m_{\{2\}} = -36:1252$ $m_{\{3\}}' = 516:2449$

$m_{\{4\}} = -62:2483$ $m_{\{5\}}' = 831:5844$

Using the table of Tchebychef polynomials ,n=14 ,we have:-

	-:574664	:235279	:0064646	:00012314	-00001686	-00000308
-8:0453	+1		-48		+540	
107:0521		+2		-96		+1080
-36:1252			+6		-225	
516:2449				+20		-630
-62:2483					*70	
831:5844						+225
$S\ T_r^2(x)$	14	910	26208	388960	3403400	19046700

RESIDUAL TABLE

	$a_i S\ R\ T_i(x)$	U^2=56:22875 S^2	$\frac{S^2}{n-i-1}$
$a_0 = \frac{-8:0453}{14}$ =-:574,664	4:623,34	51:605,41	
$a_1 = \frac{214:1042}{910}$ =:235,279	50:374,222	1:231,19	:1026
$a_2 = \frac{169:4232}{26208}$ =:006,464,6	1:095,25	<u>0:135,94</u>	<u>:0124</u>
$a_3 = \frac{47:8964}{388960}$ =:000,123,14	0:005,90	0:130,04	:0130
$a_4 = \frac{-57:3673}{3403400}$ =-:000,016,86	0:000,97	0:129,07	:0143
$a_5 = \frac{-58:7502}{19046700}$ =-:000,003,08	0:000,18	0:128,89	:0161

By considering this residual table, an interesting point comes out of this example. The effect on S^2 of the addition of three terms after the parabolic transformation is:00705 which is very slight. In other words, the quintic transformation is little, if any better than the parabolic, especially in view of the loss of three degrees of freedom. For this reason it is clear that the parabolic transformation will lead to as good a result as any obtained by including further terms. The value of $\frac{S^2}{n-1-i}$ which increases for terms after a_2 is significant in this respect. As the data consists of some fifteen class intervals and the parabolic transformation involves three constants only, it is suggested in advance that the fit will not be very close. Using a_o, a_1, a_2 , we find from the table on the previous page,

$V_o = -:884,964,8$ $\delta W = :470558$ $\delta^2 V_o = :0387876$

Building two half tables of differences for V and W ,

V	&V	δ^2V	W	&W
-:884,964,8		:038,7876	:235,279	
	:038,787,6			:470,558
-:846,177,2		:038,7876	:705,837	
	:077,575,2			:470,558
-:768,602,0		:038,7876	1:176,395	
	:166,362,8			:470,558
-:652,239,2		:038,7876	1:646,953	
	:155,150,4			:470,558
-:497,088,8		:038.7876	2:117,511	
	:193,938,0			:470,558
-:303,150,8.		:038.7876	2:588,069	
	:232,725,6			:470,558
-:070,425,2			3:058,627	

and then adding and subtracting corresponding values of V and W, we find R computed.

Th e transformation is :-

$$R(X) = -:884,965 + :470,558X + :019,393,8(X^2 - \tfrac{1}{4})$$

and $R'(X) = :470,558 + :0193938(2X)$

with

$$y = 1/\sqrt{\Pi} \;\; (:470558 + :0193938(2x))e^{-\frac{1}{2}R^2(x)}$$

The ordinates of the curve for graphing purposes are as follows-

x=	-6:5	-5:5	-4:5	-3:5	-2:5	-1:5	-0:5
2162y=	1:42	3:39	8:32	20:41	48:62	106:64	207:88

x=	+0:5	+1:5	+2:5	+3:5	+5:5	+5:5	+6:5
2162y=	342:39	451:62	450:43	318:8	149:65	43:32	7:03

The maximum ordinate occurs at $-13/2 \leqslant x \leqslant +13/2$ where

$$R''(x) = R(x)\, R'(x)^2$$

i.e.

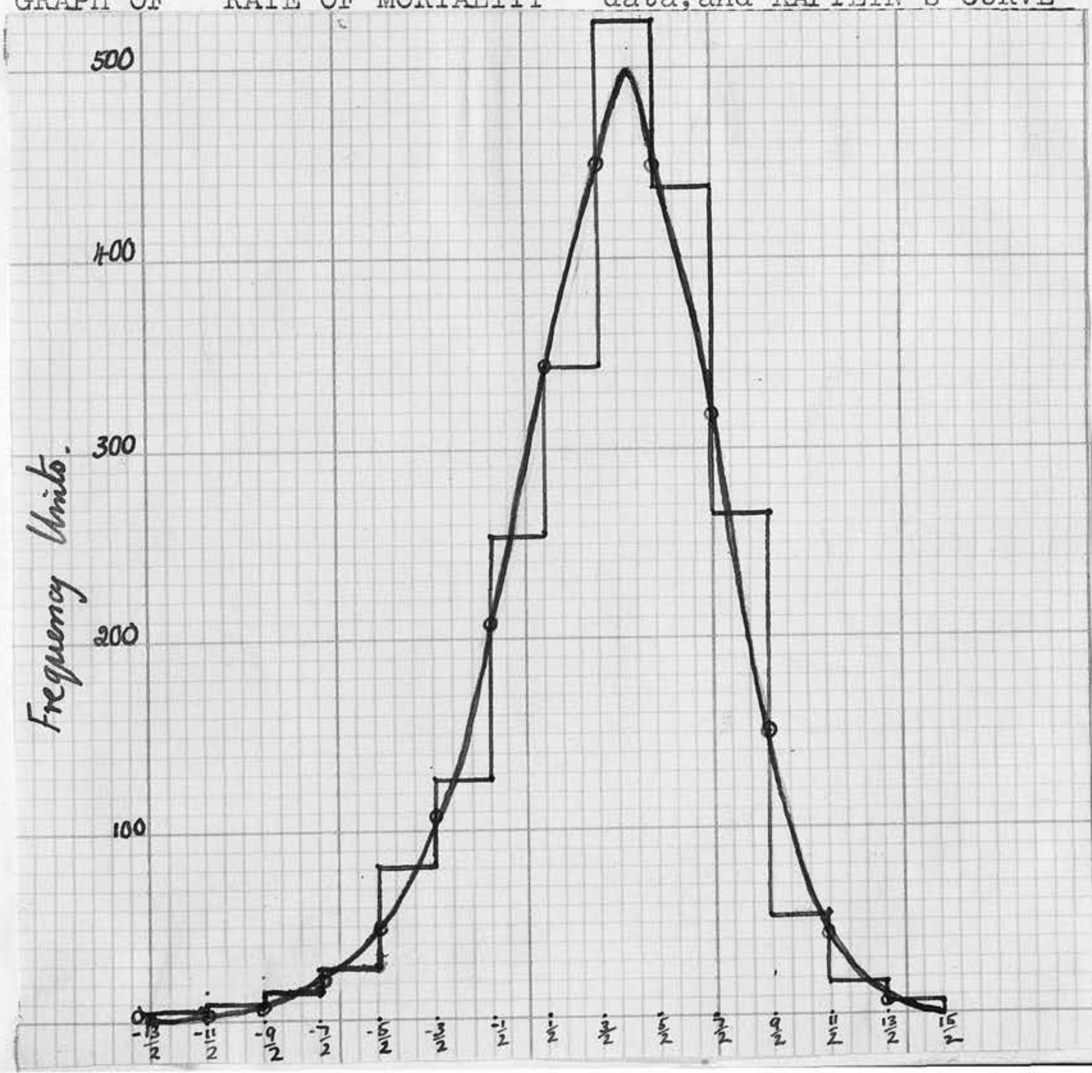
$$:0387876 = (:470558 + :0193938(2x))^2 (-:884965 + :470558x + :0193938(x^2 - \tfrac{1}{4}))$$

which is a quartic in x. This demonstrates that in the case of the very simplest Kapteyn transformation, $R(x) = a + bx + cx^2$, the problem of finding the mode is difficult.

R computed	$(R_c - R_a)^2$	Decimal	Frequency	Observed	Pearson's Type V
-3:1291	:032,22	0009	2	1	1
-2:8912	:014,69	0019	2	5	3
-2:6146	:016,80	0045	6	8	6
-2:2992	:001,81	0107	13	12	14
-1:9450	:000,23	0259	33	28	32
-1:5520	:000,45	0604	75	82	68
-1:1202	:001,96	1314	154	128	137
-0:6497	:003,56	2579	273	253	247
-0:1403	:014,40	4442	403	342	381
+0:4078	:002,41	6583	462	525	480
0:9947	:000,12	8400	393	438	441
1:6204	:038,14	9474	232	265	261
2:2849	:001,23	9888	90	53	80
2;9882	:007,78	9986	21	18	10
			3	4	1
	:135,80				

We see from the check column $(R_c - R_a)^2$ that the sum of these is in good agreement with S^2=:135,94. The graduation is not good and is without doubt inferior to Pearson's Type V . However it must be remarked that a curve involving only three constants would be remarkable if it gave a good approximation to a set of fifteen data. The fit could be improved by excluding some of the small frequencies at the tails of the distribution, for these correspond to the larger values of R which in turn account for a correspondingly large proportion of S^2, thus reducing the precision of the smaller important R's.

GRAPH OF " RATE OF MORTALITY " data, and KAPTEYN's CURVE



Example 5 Data:- "Chances of Death" Pearson :Elderton
Frequency Curves, page 135.

This example is different from the four previous examples in that it has one abrupt tail .We shall see that a Kapteyn transformed normal curve can successfully graduate the data although this of distribution calls for a modification of the method alrady set out.

x	Freq.	First Sums.	Decimal	R	R^2
0	64	64	0640	-1:0761	1:157,991
1	116	180	1799	-0:6476	:419,386
2	140	320	3199	-0:3310	:109,561
3	145	465	4648	-0:0625	:003,906
4	134	599	5988	0:1769	:031,294
5	106	705	7047	0:3804	:144,704
6	82	787	7867	0:5622	:316,069
7	72	859	8587	0:7597	:577,144
8	49	908	9076	0:9378	:879,469
9	37	945	9446	1:1275	1:271,256
10	25	970	9696	1:3260	1:758,276
11	13	983	9826	1:4929	2:228,750
12	10	993	9926	1:7233	2:969,763
13	5	998	9976	1:9900	3:960,100
14	2	1000	9996	2:3740	5:635,876
15	0:4	1000:4			21 :463,545
	1000:4				

(N.B. Tables of $1/\sqrt{\pi}\int_{-\infty}^{R} e^{-t^2}dt$ are used from Kapteyn's paper.)

FACTORIAL MOMENTS OF R

R	m(o)	$m(_1)$	$m(_2)$	$m(_3)$	$m(_4)$	$m(_5)$
-1:0761	10:7335					
-0:6476	11:8096	136:9210				
-0:3310	12:4572	125:1114	722:3533			
-0:0625	12:7882	112:6542	597:2419	2407:3599		
0:1769	12:8507	99:8660	484:5877	1810:1180	5668:1869	
0:3804	12:6738	87:0153	384:7217	1325:5303	3858:0689	9916:8925
0:5622	12:2934	74:3415	297:7064	940:8086	2532:5386	6058:8236
0:7597	11:7312	62:0481	223:3649	643:1022	1591:7300	3526:2850
0:9378	10:9715	50:3169	161:3168	419:7373	948:6278	1934:5550
1:1275	10:0337	39:3454	110:9999	258:4205	528:8905	985:9272
1:3260	8:9062	29:3117	71:6545	147:4206	270:4700	457:0367
1:4929	7:5802	20:4055	42:3428	75:7661	123:0494	186:5667
1:7233	6:0873	12:8253	21:9373	33:4233	47:2833	63:5173
1:9900	4:3640	6:7380	9:1120	11:4860	13:8600	16:2340
2:3740	2:3740	2:3740	2:3740	2:3740	2:3740	2:3740

We have used ordinary factorial moments in this example ;it is clear that it involves more computation than central moments.

Table of Tchebychef Polynomials for n=15

	:715567	:220666	-:000782	:00174183	:00000255	
10:7335	+1	-7	+91	-91	1001	-2002
136:9210		+1	-39	+78	-1430	+4290
722:3533			+6	-30	+990	-4620
2407:3599				+5	-385	+3080
5668:1869					+70	-1260
9916:8925						+252
S $T_r^2(x)$	15	280	37128	39780	6466460	42325920

Residual Table

	a_i SumRT$_i(x)$	S^2	$S^2/(n-i-1)$
$a_0 = \frac{10:7335}{15}$ =:715,567	7:680,538	13:783,007	0:9845
$a_1 = \frac{61:7865}{280}$ =:220,666	13:634,180	0:148,827	0:0114
$a_2 = \frac{-29:0507}{37128}$ =-:000,782	0:022,718	0:126,109	0:0105
$a_3 = \frac{69:29}{39780}$ =:001,741,83	0:120,691	0:005,418	0:00050
$a_4 = \frac{16:492}{6466460}$ =:000,002,55	0:000,042	0:005,376	0:00053

Inspection of this table shows that the parabolic transformation is a very slight improvement on the linear ,whereas the cubic is a pronounced improvement.The value of $S^2/(n-i-1)$ is greater for the term a_4 than a_3 and S^2 itself shows extremely little difference.Hence we decide to use the cubic transformation.

From the above table we have ,

U_0=-1:058,764 ΔU_0=:387,027 $\Delta^2 U_0$=-:056,947 $\Delta^3 U_0$=:008,709

and from these a table for R is built up thus:-

Difference Table for R

x	R	R	^{2} R	^{3}R
0	-1:058,764			
1	-0:671,737	:387,027		
2	-0:341,657	:330,080	-:056,947	
3	-0:059,815	:281,842	-:048,238	:008,709
4	+0:182,498	:242,313	-:039,529	:008,709
5	+0:393,991	:211,493	-:030,820	:008,709
6	+0:583,373	:189,382	-:022,111	:008,709
7	+0:759,353	:175,980	-:013,402	:008,709
8	+0:930,640	:171,287	-:004,693	:008,709
9	+1:105,943	:175,303	+:004,016	:008,709
10	+1:293,971	:188,028	+:012,725	:008,709
11	+1:503,433	:209,462	+:021,434	:008,709
12	+1:743,038	:239,605	+:030,143	:008,709
13	+2:021,495	:278,457	+:038,852	:008,709
14	+2:347,513	:326,018	+:047,561	:008,709
15	+2:729,801	:382,288	+:056,270	:008,709

For a good fit at the abrupt tail, we expect a large value of R to correspond to x=-1 . To compute values of R for values of x at the negative side , we refer to the actual transformation

$R(x)=-1{:}0587635 +{:}41840324x -{:}032828x^2 +{:}001451525x^3$

and find $R(-1)= -1{:}5114$
$R(-2)=-2\ {:}0385$
$R(-3)= -2{:}6486$

We can see immediately that the graduated data will have frequencies at the abrupt tail for which there are no corresponding data. This defect is unavoidable with a transformation expressed as a power series for it is clearly impossible for such an expression to have an 'infinity' for a finite value of x . We shall return to this point at the end of this example.

The computed values of R leading to the graduated data are now given .

x	R computed	$(R_c-R)^2$	Decimal	Frequency	Observed
-3	-2:6486		0001		
-2	-2:0385		0020	2	
-1	-1:5114		0163	14	
0	-1:0588	:000,30	0671	51	64
+1	-0:6717	:000,58	1711	104	116
+2	-0:3417	:000,11	3145	143	140
+3	-0:0598	:000,01	4663	152	145
+4	0:1825	:000,03	6018	136	134
+5	0:3940	:000,18	7113	110	106
+6	0:5834	:000,45	7956	84	82
+7	0:7594	:000,00	8586	63	72
+8	0:9306	:000,05	9059	47	49
+9	1:1059	:000,47	9411	35	37
+10	1:2940	:001,02	9663	25	25
+11	1:5034	:000,11	9833	17	13
+12	1:7430	:000,39	9932	10	10
+13	2:0215	:000,99	9979	5	5
+14	2:3475	:000,70	9995	2	2
+15	2:7298		9999		0:4
		:005,39			

i n good agreement with S^2=:oo5,418

COMPARISON BETWEEN SEVERAL GRADUATIONS OF THIS DATA .

OBSERVED	PEARSON TYPE 1	KAPTEYN	TYPE B	EDGEWORTH
				1
		2		9
	2	14	12	30
64	67	51	64	64
116	116	104	104	102
140	138	143	129	130
145	139	152	134	135
134	128	136	128	130
106	110	110	116	111
82	89	84	93	92
72	69	63	73	73
49	51	47	53	53
37	35	35	36	36
25	24	25	25	20
13	15	17	14	10
10	9	10	10	10
5	5	5	5	4
2	2	2	2	-
0:4	1	0:4	1	-

Each of the graduations depends on four constants and so a direct comparison is possible . Without applying the χ^2 test, it is evident that the Kapteyn graduation is superior to Type B and Edgeworth. The comparison between Kapteyn's and Pearson's Type 1 shows a close agreement except in the case of the frequencies 14 and 2 in the former which have no counterpart in the data. This represents 1:6% of the total frequency and hence it can not be regarded as a serious defect. However we may expect that some distributions with one abrupt tail will present difficulties if we attempt a graduation by means of a Kapteyn transformation of the type

$$R(x) = a + bx + cx^2 + dx^3 + \dots\dots$$

For suppose $x=0$ corresponds to the frequency at the abrupt tail. Then the ordinate of the curve at $x=-1$ must be zero (or at some point $-1 \leqslant x \leqslant 0$). $R(-1)$ must thus be $-\infty$ which is impossible with the form $R(x)$ used above. Common sense would suggest a transformation of the 'Laurent' series type , i.e.

$$R(x) = a + bx + cx^2 + dx^3 + \dots\dots\dots$$
$$+Ax^{-1} + Bx^{-2} + Cx^{-3} + \dots\dots\dots .$$

but this would lead to equations to determine a,b,c,..... A,B,C,.... which can not be solved . In the next example we shall discuss a transformation which can be treated practically and which also makes $R(-a) = -\infty$.

We pass on to the application of the χ^2 test to the Kapteyn and Pearson graduations.

The Value of P_{χ^2} for the Kapteyn and Pearson Curves.

Observed	Kapteyn	Pearson	$(K-\theta)^2/K$	$(P-0)^2/P$
	2			
	14	2		
64	51	67	:13	:36
116	104	116	1:38	:00
140	143	138	:07	:03
145	152	139	:32	:26
134	136	128	:03	:28
106	110	110	:15	:15
82	84	89	:05	:55
72	63	69	1:12	:13
49	47	51	:08	:08
37	35	35	:11	:11
25	25	24	:00	:04
13	17	15	:94	:27
10	10	9		
5	5	5		
2	2	2		
			4:38	2:26

The Arrows indicate the class grouping of the application of the χ^2 test.

n= 13-4=9 = number of degrees of freedom ,in both cases.

P (Kapteyn) = 0:88 P(Pearson) = 0:986

The value for the Kapteyn curve shows good agreement while P for Pearson's curve is so high as to be doubtful. In conclusion we can say that the Kapteyn curve gives a reasonable account of the data.

EQUATION OF THE CURVE AND VALUE OF ORDINATES.

We have $R(x) = -1{:}0587635 + {:}418403x - {:}032828x^2 + {:}001451525x^3$ and so

$$R'(x) = {:}418403 - {:}065656x + {:}004354575x^2$$

and the equation of the curve is

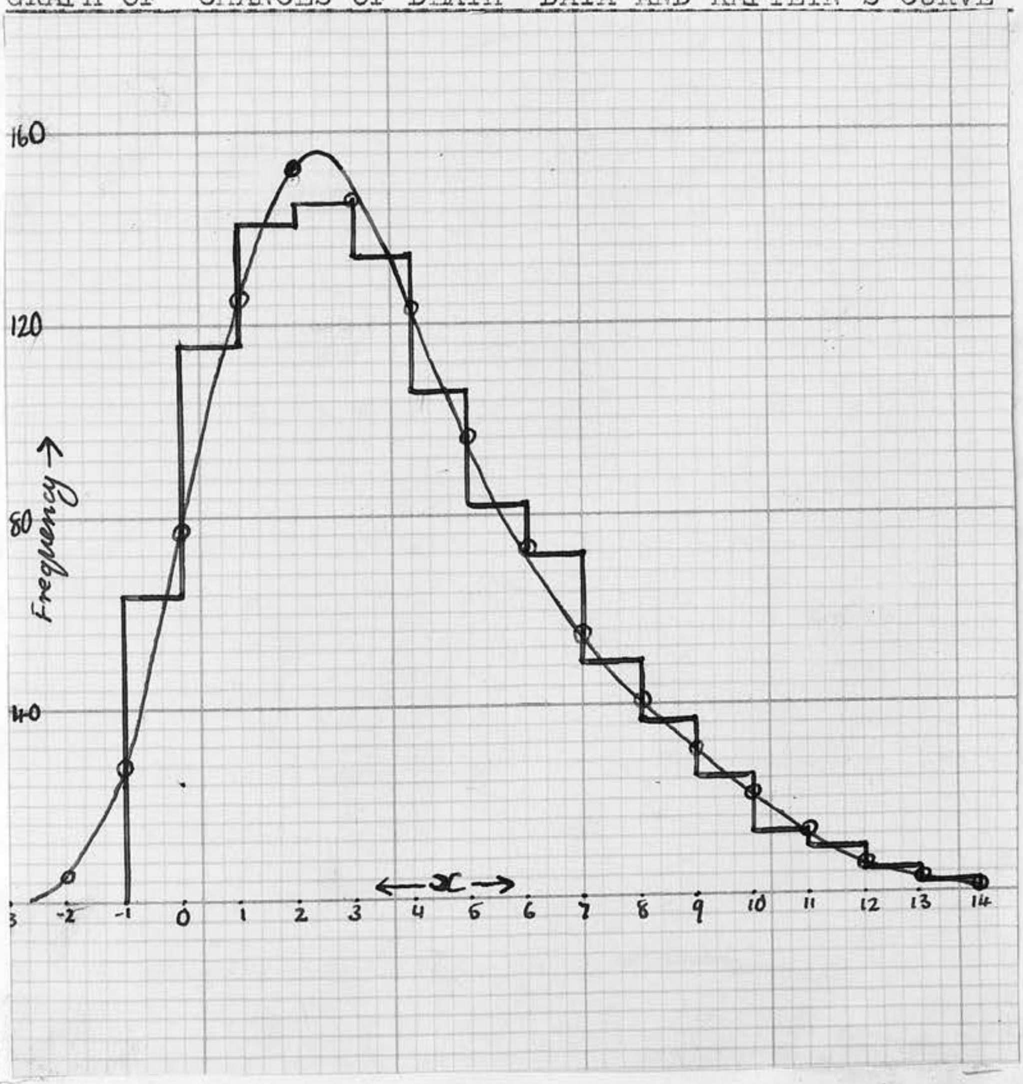
$$y = \frac{({:}418403 - {:}065656x + {:}004354575x^2)}{ii} e^{-(R(x))^2}$$

The values of y were found to be :-

x=	-2	-1	0	1	2	3	4	5	6	7
y=	·0050	·0281	·0769	·1252	·1529	·1465	·1231	·0961	·0727	·0546

x=	8	9	10	11	12	13	14
y=	·0408	·0299	·0209	·0131	·0070	·0029	·0008

GRAPH OF "CHANCES OF DEATH" DATA AND KAPTEYN'S CURVE



4B FURTHER EXAMPLES OF KAPTEYN'S IDEA OF A TRANSFORMED VARIABLE, WHEN THE TRANSFORMATION IS LOGARITHMIC.

The case already considered is $y=R'(x)e^{-R^2(x)}$
where $R(x) = a_oT_o(x) + a_1T_1(x) + a_2T_2(x) + \ldots\ldots\ldots\ldots\ldots$
and the $T_r(x)$'s are Tchebychef's orthogonal polynomials .
In the present case we assume

$$hR(x) = \log_e(a_oT_o(x) + a_1T_1(x) + a_2T_2(x) + \ldots\ldots\ldots)$$

i.e. $e^{hR(x)}= a_oT_o(x) + a_1T_1(x) + a_2T_2(x) + \ldots\ldots\ldots\ldots\ldots$
where it is understood that h is positive and the constants a_i, i=1,2,3,4,5, etc. are chosen so that

$$a_oT_o(0) + a_1T_1(0) + a_2T_2(0) + \ldots\ldots\ldots =0$$

approximately, in which case $e^{hR(0)} = 0$ and so since h is positive $R(0) = -$. In this way the transformation will be suitable for distributions with one (or two) abrupt tail(s).
The special case $e^{R(X)}= a_oT_o(X) + a_1T_1(X)$, has been considered by Wicksell in "On the Genetic Theory of Frequency " (ref. 4), where he uses the method of moments . It is clear however that this method would not apply to parabolic, cubic, ...etc. forms of $e^{hR(X)}$. The method set out here is similar to that explained in 4A except that a special artifice is used in the case when h in $e^{hR(X)}$ is $\neq$ 1.

EXAMPLE 1

First of all we take the simple case $e^{R(X)}=a_o+a_1T_1(X) + \ldots$
i.e. h=1, and give a successful graduation of the data treated in Example 5, Pearson's "Chances of Death" distribution.

Distribution 64 116 140 145 134 106 82 72 49 37 25 13
10 5 2 0:4

We find R as previously explained, and then using a table of natural logarithms (such as those in 'Fisher and Yates' "Statistical Tables") write down e^R and graduate by means of Tchebychef polynomials. As x=0 corresponds to the frequency 64, we include in the data frequency 0 corresponding to x=-1 . If we excluded this frequency (which is a perfectly legitimate device) there would still be no check on the spreading out of the abrupt tail.

x	Frequency	First Sums	R	e^R	$(e^R)^2$
0	0	0	–	0	0
1	64	64	-1:5223	0:2182	0:0476
2	116	180	-0:9158	0:4002	0:1602
3	140	320	-0:4681	0:6262	0:3921
4	145	465	-0:0882	0:9156	0:8383
5	134	599	0:2503	1:2844	1:6497
6	106	705	0:5380	1:7126	2:9330
7	82	787	0:7952	2:2149	4:9058
8	72	859	1:0745	2:9285	8:5761
9	49	908	1:3262	3:7667	14:1880
10	37	945	1:5945		33:6908
11	25	970	1:8750		
12	13	983	2:1111		
13	10	993	2:4360		
14	5	998	2:8200		
15	2	1000	3:3657		
16	0:4	1000:4			

We have taken x=0 to correspond to frequency zero . Further the data has been curtailed at x=9 , for values of e^R for x= 10,11,... 16 become rapidly large and would disturb the precision for the smaller values of x which correspond to large frequencies.

We now graduate e^R and set out the Central Moments and Residual Tables.

CENTRAL MOMENTS OF e^R

e^R	$u\hat{\Sigma}_1$	$\hat{\Sigma}_1^{\prime 2}$	$u\hat{\Sigma}_1^{\prime 3}$	$\hat{\Sigma}_1^{4}$	$u\hat{\Sigma}_1^{\prime 5}$
0	0	0	0	0	0
0:2182	0:2182	0:2182	0:2182	0:2182	0:2182
0:4002	0:6184	0:8366	1:0548	1:2730	1:4912
0:6262	1:2446	2:0812	3:1360	(2:8410)	
0:9156	2:1602	(3:1613)			
1:2844	11:9071	(35:94825)			
1:7126	10:6227	29:9947	63:5953	(83:39355)	
2:2149	8:9101	19:3720	33:6006	51:5959	73:3579
2:9285	6:6952	10:4619	14:2286	17:9953	21:7620
3:7667	3:7667	3:7667	3:7667	3:7667	3:7667

From this we have :- $m_{\{0\}}$ =14:0673 $m_{\{1\}}'$ = 32:78695

$m_{\{2\}}$ =66:7313 $m_{\{3\}}'$=80:55255 $m_{\{4\}}$ =74:8491

CENTRAL POLYNOMIAL TABLE FOR n= 10

$a_r \rightarrow$ / $m_r \downarrow$	1:40673	0:19871	0:01321	0:00109	0:00004
14:0673	+1		-24		+126
32:78695		+2		-48	
66:7313			+6		-105
80:55255				+20	
74:8491					+70
$S\ T_r^2(x)$	10	330	4752	34320	140140

RESIDUAL TABLE

		$a_i S\ RT_i(x)$	S^2	$S^2/(n-1-i)$
$a_0 = \frac{14:0673}{10}$	=1:40673	19:7889	13:9019	1:5
$a_1 = \frac{65:5739}{330}$	=0:19871	13:0302	0:8717	0:1
$a_2 = \frac{62:7726}{4752}$	=0:01321	0:8292	0:0425	0:006
$a_3 = \frac{37:2774}{34320}$	=0:00109	0:0406	0:0019	0:0003
$a_4 = \frac{5:1303}{140140}$	=0:00004	0:0002	0:0017	0:0003

From the residual table it is clear that the cubic transformation is to be chosen. Further we have :-

μV_0 =1:08969 &W_0=0:34510 μ&$^2V_0$=0:07926 &$^3W_0$=0:02180

Using these we build up two half tables of differences for V and and W which compounded lead to e^R

V	&V	&^{2}V	W	&W	&^{2}W	&^{3}W
				0:34510		0:02180
1:08969		0:07926	0:17255		0:01090	
	:07926			0:35600		0:02180
1:16895		0:07926	0:52855		0:03270	
	:15852			0:38870		0:02180
1:32747		0:07926	0:91725		0:05450	
	:23778			0:44320		0:02180
1:56525		0:07926	1:36045		0:07630	
	:31704			0:51950		0:02180
1:88229		0:07926	1:87995		0:09810	
	:39630			0:61760		0:02180
2:27859		0:07926	2:49755		0:11990	
	:47556			0:73750		0:02180
2:75415		0:07926	3:23505		0:14170	
	:55482			0:87920		0:02180
3:30897		0:07926	4:11425		0:16350	
	:63408			1:04270		0:02180
3:94305		0:07926	5:15695		0:18530	
	:71334			1:22800		0:02180
4:65639		0:07926	6:38495		0:20710	
	:79260			1:43510		0:02180
5:44899		0:07926	7:82005		0:22890	
	:87186			1:66400		0:02180
6:32085		0:07926	9:48405		0:25070	
	:95112			1:91470		
7:27197			11:39875			

We find the computed values of e^R to be as follows:-

e^R	e^R (actual)	Difference	(Diff.)^2
0:00234	0	-0:00234	:000,005
0:20480	0:2182	+0:01340	:000,180
0:41022	0:4002	-0:01002	:000,100
0:64040	0:6262	-0:01420	:000,202
0:91714	0:9156	-0:00154	:000,002
1:26224	1:2844	+0:02216	:000,491
1:69750	1:7126	+0:01510	:000,228
2:24472	2:2149	-0:02982	:000,889
2:92570	2:9285	+0:00280	:000,008
3:76224	3:7667	+0:00446	:000,020
			:002,125

The value of the sum of the squares of the differences $\left(e^{R}\text{ actual} - e^{R}\text{ computed}\right)$ are found to be 0:002125 in good agreement with S^2=0:0019 . We proceed to the computation of the frequency .

e^R graduated	R	Decimal	Frequency	Observed
0:00234	-6:0576	0000		
0:020480	-1:5857	0564	56	64
0:41022	-0:8911	1864	130	116
0:64040	-0:4457	3279	142	140
0:91714	-0:0865	4655	138	145
1:26224	0:2328	5921	127	134
1:69750	0:5292	7016	110	106
2:24472	0:8086	7908	89	82
2:92570	1:0735	8585	68	72
3:76224	1:3250	9074	49	49
4:77614	1:5636	9410	34	37
5:98920	1:7900	9633	22	25
7:42322	2:0046	9775	14	13
9:10000	2:2083	9864	9	10
11:04134	2:4016	9918	5	5
13:26904	2:5848	9951	3	2
15:80490	2:7603	9971	2	0:4
18:67072	2:9269	9983	1	

For this graduation χ^2 =5:59 with the same grouping as used before in example 5 page 33 ,and since n (number of degrees of freedom) =9 , we have P_{χ^2}=0:78 approx. which indicates that the logarithmic Kapteyn curve accounts for the whole of the facts. It is to be recalled that the value of χ^2 for the ordinary Kapteyn transformation was 4:38 with P_{χ^2}= 0:88 . However the above graduation does give a curve with an abrupt tail .

EQUATION OF THE CURVE

From the table of Tchebychef polynomials on page 37, we have

$$e^{R(X)} = :00234 + :20824\dot{6}X - :00942X^2 + :00363X^3$$

and hence $e^R \frac{dR}{dx} = :20824\dot{6} - :01884x + :0108\dot{9}x^2$

The Equation of the curve is :-

$$y=1/\sqrt{2\pi}\; e^{-R^2(x)} \quad (:20824\dot{6} - :01884x + :01089x^2)\; e^{-R(x)}$$

$$= \sqrt{\frac{e}{2\pi}}\; (:20824\dot{6} - :01884x + :01089x^2)\; e^{-\frac{1}{2}(R + 1)^2}$$

$$= \left(1/\sqrt{2\pi}\right) e^{-\frac{1}{2}(R + 1)^2} \quad E(x) \quad \text{where } E(x) = (:20824\dot{6} - :01884x + :01089x^2)\sqrt{e}$$

The following scheme shows the calculation of ordinates:-

x	$\lvert R + 1\rvert$	$(1/\sqrt{2\pi})\; e^{-\frac{1}{2}(R+1)^2}$	$E(x)$	y
0	-	0	-	0
1	:5857	:3362	:3302	:1110
2	:1089	:3966	:3531	:1400
3	:5543	:3429	:4119	:1412
4	:9135	:2629	:5066	:1331
5	1:2328	:1865	:6373	:1189
6	1:5292	:1241	:8039	:0998
7	1:8086	:0779	1:0065	:0784
8	2:0735	:0465	1:2450	:0579
9	2:3250	:0267	1:5196	:0406
10	2:5636	:0150	1:8298	:0274
11	2:7900	:0081	2:1761	:0176
12	3:0046	;0039	2:5584	:0100
13	3:2083	:0023	2:9743	:0068
14	3:4016	:0012	3:4312	:0041 .

The last column gives the ordinates of the curve whose equation can be written

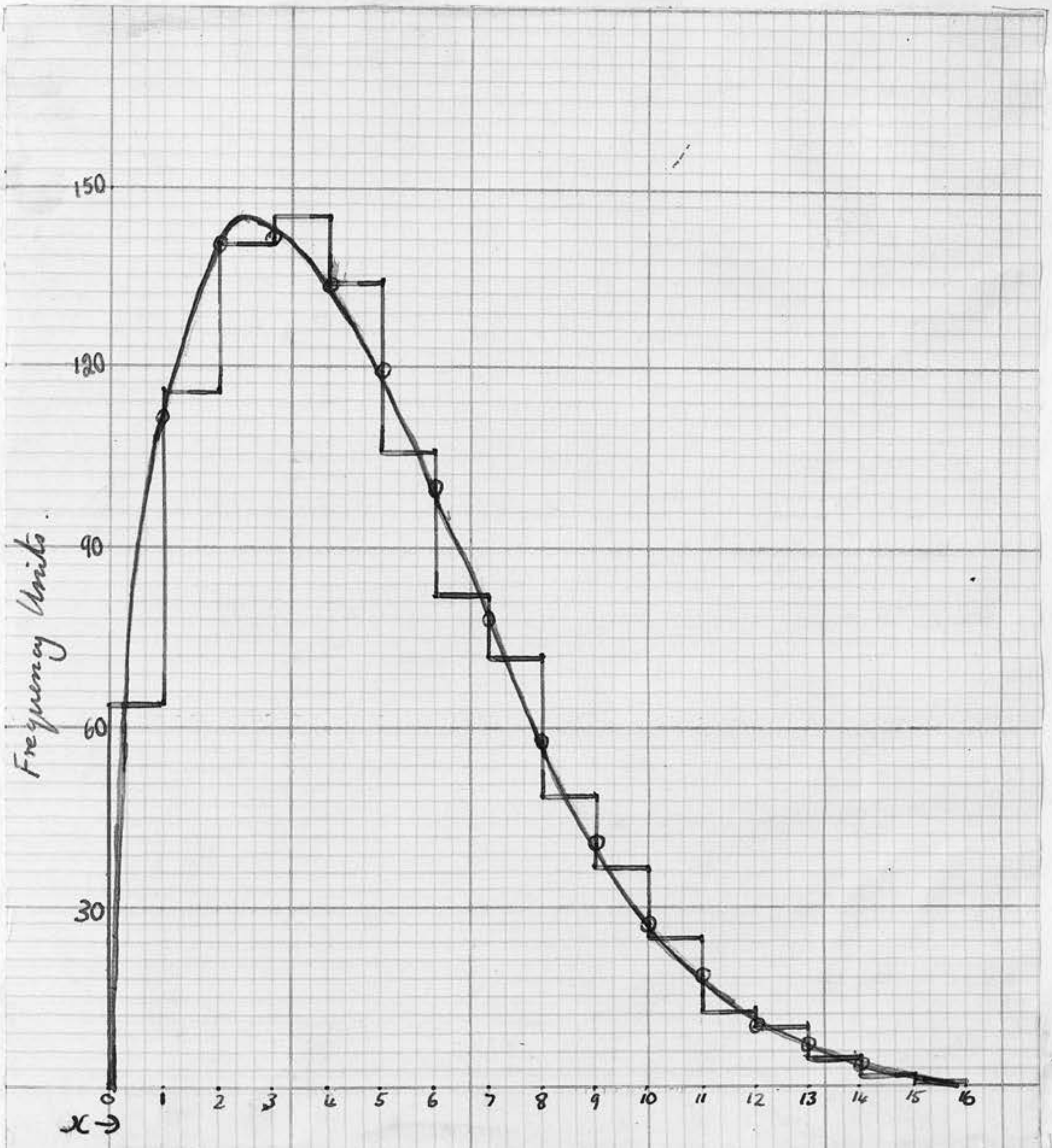
$$y= \sqrt{\frac{e}{2\pi}}(:20824\dot{6}-:01884x+:0108\dot{9}x^2)e^{-\frac{1}{2}\left[1+\log(:00234+:208246x-:00942x^2+:00363x^3)\right]^2}$$

where e= 2:71828 to five places of decimals . The graph of the curve is given overleaf.

GRAPH OF :-

PEARSON'S "CHANCES OF DEATH" data.

Kapteyn Logarithmic Transformation



Example 2 Data:- "Exposed to Risk of Sickness" Watson M.U. Tables; Elderton "Frequency Curves" page 7.

We again try the simple transformation

$e^{R(X)} = a + b\,T_1(X) + cT_2(X) +$ ..... and proceed as in example 1.

Freq.	First Sums	R	e^R	$(e^R)^2$
			0	0
34	34	-1:8250	0:1612	:025,985
145	179	-0:9192	0:3988	:159,041
156	335	-0:4261	0:6531	:426,540
145	480	-0:0503	0:9510	:904,401
123	603	0:2611	1:2982	1:685,323
103	706	0:5418	1:7191	2:955,305
86	792	0:8134	2:2555	5:087,280
71	863	1:0941	2:9864	8:918,585
55	918	1:3920	4:0229	16:183,724
37	955	1:6956	5:4499	29:701,410
21	976	1:9780		66:047,594
13	989	2:2900		
7	996	2:6500		
3	999			
1	1000			

(Frequency 0 is inserted at the beginning of the data.)

We graduate the data using the frequencies up to and including the 37 group.

CENTRAL MOMENTS OF e^R

e^R	$\mu\hat{\Sigma}$	$\hat{\Sigma}^{\prime 2}$	$\mu\hat{\Sigma}^3$	$\hat{\Sigma}^4$	$\mu\hat{\Sigma}^5$
0	0	0	0	0	0
0:1612	0:1612	0:1612	0:1612	0:1612	0:1612
0:3988	0:5600	0:7212	0:8824	1:0436	1:2048
0:6531	1:2131	1:9343	2:8167	3:8603	(3:13495)
0:9510	2:1641	4:0984	(4:8659)		
1:2982	(2:8132) (17:0829)				
1:7191	16:4338	58:5304	(119:1163)		
2:2555	14:7147	42;0966	89:8511	163:4281	(186:56345)
2:9864	12:4592	27:3819	47:7545	73:5770	104:8494
4:0229	9:4728	14:9227	20:3726	25:8225	31:2724
5:4499	5:4499	5:4499	5:4499	5:4499	5:4499

$m_{\{0\}}' = 19:8961$ $m_{\{1\}} = 54:4320$ $m_{\{2\}}' = 123:9822$

$m_{\{3\}} = 159:5678$ $m_{\{4\}}' = 189:6984$

Inserting the moments in the Tchebychef Table for n=11, we have:-

a_r) →	1:80874	:247418	:019038	:002086	:000284
m_r ↓					
19:8961	+1		−30		+210
54:4320		+2		−56	
123:9822			+6		−140
159:5678				+20	
18 9:6984					+70
$S\ T_r^2(x)$	11	440	7722	68640	350350

and the values of a and S^2 are as follows:-

	$a_i S\ RT_i(x)$	S^2	$\frac{S^2}{n-1-i}$
		=66:047594	
$a_0 = \frac{19:8961}{11}$ =1:80874	35:986,872	30:060722	3:006
$a_1 = \frac{108:864}{440}$ =0:247418	26:934,913	3:125809	0:35
$a_2 = \frac{147:0102}{7722}$ =0:019038	2:798780	0:327029	0:04
$a_3 = \frac{143:164}{68640}$ =0:002086	0:298640	0:028389	0:004
$a_4 = \frac{99:5610}{350350}$ =0:000284	0:028275	0:000114	0:00002

The quartic transformation is clearly a good fit as the residual after a_4 is 0:000114 and the initial value of S^2 is 66:047594 . Using this quartic expression we proceed to find V and W and from them e^R . From the table at the top of this page we have :-

V_0=1:297240 $\mu\delta W_0$=0:378020 $\delta^2 V_0$=0:074468 $\mu\delta^3 W_0$=0:041720

$\delta^4 V_0$=0:019880

V	&V	&^{2}V	&^{3}V	&^{4}V
1:297,240		:074,468		:019,880
	:037,234		:009,940	
1:334,474		:084,408		:019,880
	:121,642		:029,820	
1:456,116		:114,228		:019,880
	:235,870		:049,700	
1:691,986		:163,928		:019,880
	:399,798		:069,580	
2:091,784		:233,508		:019,880
	:633,306		:089,460	
2:725,090		:322,968		:019,880
	:956,274		:109,340	
3:681,364		:432,308		:019,880
	1:388,582		:129,220	
5:069,946		:561,528		:019,880
	1:950,110		:149,100	
7:020,056		:710,628		:019,880
	2:660,738		:168,980	
9:680,794		:879,608		
	3:540,346			
13:221,140				

W	&W	&^{2}W	&^{3}W
0		0	
	:378,020		:041,720
0:378,020		0:041,720	
	:419,740		:041,720
0:797,760		0:083,440	
	:503,180		:041,720
1:300,940		0:125,160	
	:628,340		:041,720
1:929,280		0:166,880	
	:795,220		:041,720
2:724,500		0:208,600	
	1:003,820		:041,720
3:728,320		0:250,320	
	1:254,140		:041,720
4:982,460		0:292,040	
	1:546,180		:041,720
6:528,640		0:333,760	
	1:879,940		:041,720
8:408,580		0:375,480	
	2:255,420		
10:664,000			

Combining corresponding values of V and W the graduated values of e^{R} are found.

e^R Compt.	e^R Actual	$(\text{Diff.})^2$	R	Decimal
:000,590	0	:000,000	-7:4354	0000
:162,504	0:161,200	:000,002	-1:8171	0346
:391,046	0:398,800	:000,060	-0:9389	1739
:658,356	0:653,100	:000,028	-0:4180	3379
:956,454	0:951,000	:000,030	-0:0445	4822
1:297,240	1:298,200	:000,001	0:2603	6027
1:712,494	1:719,100	:000,044	0:5379	7047
2:253,876	2:255,500	:000,002	0:8127	7918
2:992,926	2:986,400	:000,043	1:0963	8635
4:021,064	4:022,900	:000,003	1:3915	9179
5:449,590	5:449,900	:000,000	1:6955	9550
7:409,684		:000,213	2:0028	9774
10:052,406			2:3077	9895
13:548,696			2:6062	9954
18:089,374			2:8953	9981
23:885,140			3:1733	9992

The difference between S (e^Rcomputed $-e^R$actual$)^2$ and S^2 is :0001 which indicates a good degree of accuracy in the computation.

Computed Frequency	Observed	Difference	$(\text{Diff.})^2$ Frequ. Compt.
35	34	+1	:03
139	145	-6	:26
164	156	+8	:39
144	145	-1	:01
121	123	-2	:03
102	103	-1	:01
87	86	+1	:01
72	71	+1	:01
54	55	-1	:02
37	37	0	:00
22	21	+1	:05
12	13	-1	:08
6	7	0	
3	3		:90
1	1		
1			

The agreement is excessively close and the value of χ^2 =0:91 corresponds to P_{χ^2} greater than 0:99. We must view the Kapteyn logarithmic transformed curve in this case with some suspicion.

EQUATION OF THE CURVE

The equation of the transformation is

$$e^{R(X)} = 0{:}00059 + 0{:}11434\dot{3}X + 0{:}056355\dot{6}X^2 - 0{:}00961\dot{3}X^3 + 0{:}000828\dot{3}X^4$$

and so the fitted curve is

$$y = \frac{e}{\sqrt{2\pi}}\left({:}11434\dot{3} + {:}11271\dot{3}x - {:}02883\dot{9}x^2 + {:}00331\dot{3}x^3\right) e^{-\frac{1}{2}(R+1)^2}$$

where R(x) is the natural logarithm of the expression above.

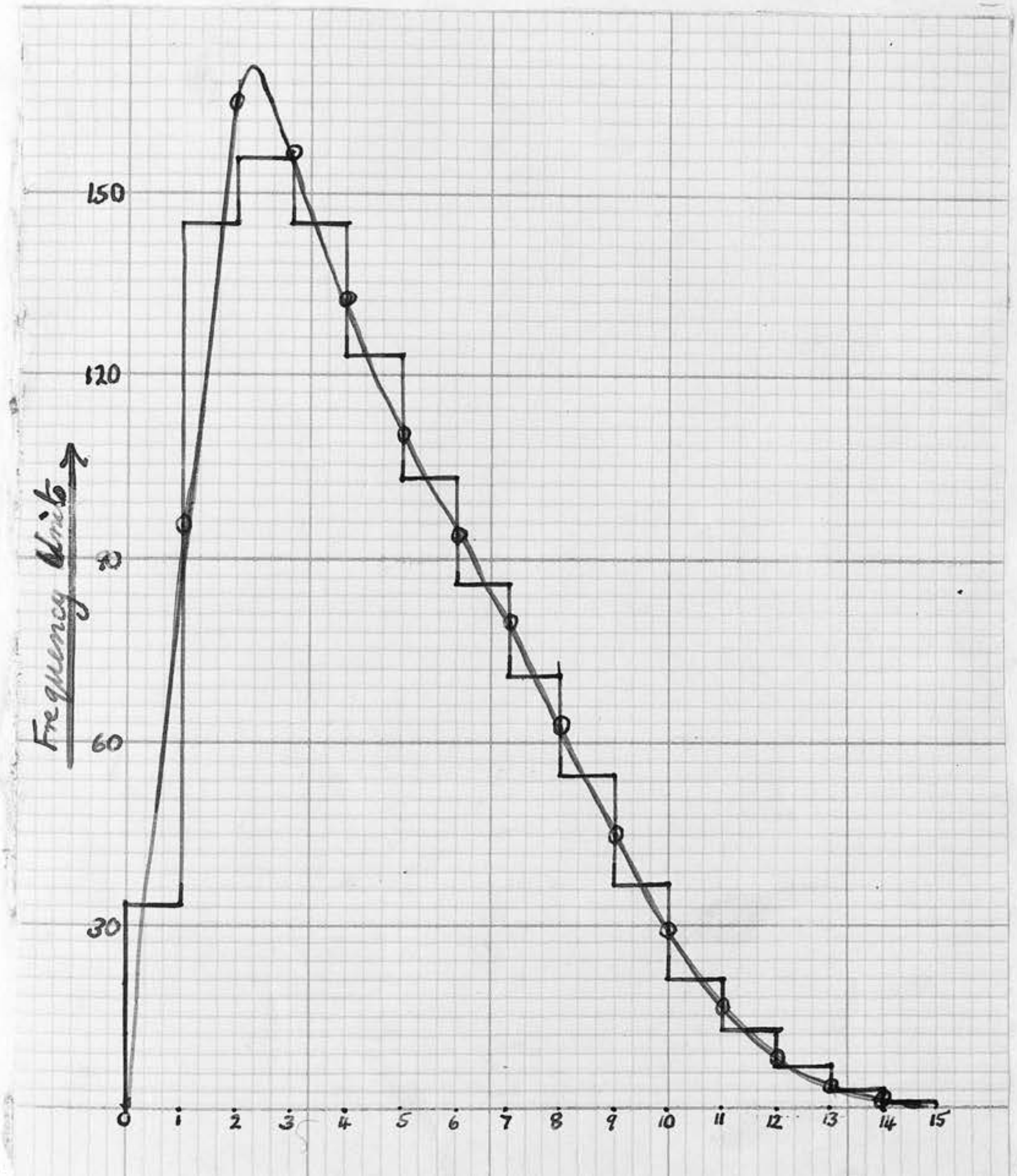
We write $y = E(x)\sqrt{\frac{1}{2\pi}}\, e^{-\frac{1}{2}(R+1)^2}$ and compute ordinates as follows:-

x=	$\sqrt{\frac{1}{2\pi}}\, e^{-\frac{1}{2}(R+1)^2}$	E(x)	y
0	0	-	0
1	0:2858	:3323	:0950
2	0:3982	:4133	:1646
3	0:3368	:4656	:1568
4	0:2528	:5207	:1316
5	0:1804	:6118	:1104
6	0:1223	:7717	:0944
7	0:0771	1:0331	:0797
8	0:0443	1:4289	:0633
9	0:0228	1:9918	:0454
10	0:0105	2:7546	:0289
11	0:0044	3:7501	:0165
12	0:0017	5:0110	:0085
13	0:0006	6:5701	:0039
14	0:0002	8:4602	:0017

The frequency diagram and graph of the curve are given overleaf.

"EXPOSED TO RISK OF SICKNESS" Data.

Kapteyn Logarithmic Curve.



Example 3 "Number of Wives tabulated for the ages of mothers and according to years since marriage." (Trans. Actua. Soc. Ed. lv 4)

x	Freq.	First Sums	Decimal	R	e^R	$(e^R)^2$
0	0	0	0000	$-\infty$	0:0000	000000
1	44	44	1753	-0:9335	:3932	:154606
2	135	179	7131	+0:5624	1:7549	3:079674
3	45	224	8924	1:2394	3:4536	11:927353
4	12	236	9402	1:5567	4:7431	22:496998
5	8	244	9721	1:9129	6:7727	45:869465
6	3	247	9841	2:1475		83:528096
7	1	248	9880	2:2567		
8	3	251				
	251					

We use the data up to and including x=5 for graduation purposes.

CENTRAL MOMENTS OF e^R

e^R	$\hat{\Sigma}$	$\mu\hat{\Sigma}^2$	$\hat{\Sigma}^3$	$\mu\hat{\Sigma}^4$
0	0	0	0	0
:3932	:3932	:3932	:3932	(:1966)
1:7549	2:1481	(1:46725)		
3:4536	14:9694	(25:7732)		
4:7431	11:5158	18:2885	25:0612	(19:3033)
6:7727	6:7727	6:7727	6:7727	6:7727

$m_{\{0\}}$ =17:1175 $m_{\{1\}}'$ =24:30595 $m_{\{2\}}$ =25:4544 $m_{\{3\}}'$ =19:1067

Using the table of Tchebychef polynomials for n=6 we find the a_r's.

	2:852917	:694456	:046983	-:009391
17:1175	+1		-8	
24:30595		+2		-16
25:4544			+6	
19:1067				+20
6:7727				
S $T_r^2(x)$	6	70	336	720

We pass on to the residual table which also shows the calculation of the a's.

RESIDUAL TABLE

	$a_i S\, RT_i(x)$	S^2	$\frac{S^2}{n-1-i}$
$a_o = \frac{17:1175}{6} = 2:852,917$	48:834,807	34:693289	7:0
$a_1 = \frac{48:6119}{70} = 0:694,456$	33:758,826	0:934463	0:23
$a_2 = \frac{15:7864}{336} = 0:046,983$	0:741,692	0:192771	0:0642
$a_3 = \frac{-6:7612}{720} = -0:009,391$	0:063,494	0:129277	0:0646

It appears that the quadratic transformation is better than the cubic,but the value of S^2 at this point,namely 0:192771 is large in comparison with the number of groups in the distribution i.e. 6. Without continuing to find e^R we can see that the fit would not be very satisfactory . We thus try a transformation

$$e^{hR(x)} = a_o + a_1T_1(x) + a_2T_2(x) + \;\ldots\ldots\;\;\ldots.$$

where we assume that h is positive and less than 1. The difficulty is to find h. Let us confine our attention to values of R for which mod.hR is less than 1. Then

$$e^{hR} = 1 + hR + \frac{h^2R^2}{2!} \quad \text{approximately.}$$

We can choose the first few values of R from the data (which will usually be of magnitude less than 1:5 for R=1:5 represents an area of approx. 90% of the whole) and proceed to determine the transformation

$$e^{hR} = a_o + a_1T_1(x) + a_2T_2(x) + \ldots\ldots\ldots\;\;\ldots$$

using $1 + hR + \frac{h^2R^2}{2!}$ in place of e^{hR}.

The a's will be determined in terms of h and the residual at any stage will be a quartic equation in h. If we decide in advance to fit a cubic transformation ,then we make the S^2 after a_3 a minimum with respect to h.This will result in a cubic equation in h from which we find the positive root less than unity ,assuming one such root exists. Returning to the data and using this value of h,we graduate e^{hR} and terminate the transformation at the most suitable point ;which may or may not be the the term representing the cubic.We apply this method to the data under consideration Consider the first four values of R :-

R= $-\infty$ -0:9335 0:5624 1:2394

e^{hR} =0 for h positive and finite and R= $-\infty$

We graduate these values of $e^{hR} = 1 + hR + \frac{h^2R^2}{2!}$

e^{hR} approx.	$m_{(o)}$	$m_{(1)}$
0	$\underline{3+:8683h+1:3618h^2}$	
$1-:9335h+:4357h^2$	$3+:8683h+1:3618h^2$	$\underline{6+3:9095h+3:0559h^2}$
$1+:5624h+:1581h^2$	$2+1:8018h+:9261h^2$	$3+3:0412h+1:6941h^2$
$1+1:2394h+:7680h^2$	$1+1:2394h+:7680h^2$	$1+1:2394h+:7680h^2$

$m_{(2)}$

$\underline{4+4:2806h+2:4621h^2}$
$1+1:2394h+:7680h^2$

A quadratic transformation is fitted to these four values and the residual is made a minimum with respect to h.

The Residual is clearly:-$S^2 =(1-:9335h+:4357h^2)^2 + (1+1:2394h+:7680h^2)^2 +(1+:5624h +:1581h^2)^2-\frac{(3+:8683h+1:3618h^2)^2}{4}$

$-\frac{(3+5:2141h+2:0264h^2)^2}{20} \quad -\frac{(-3+4:8315h+:5226h^2)^2}{36}$

The table of Tchebychef polynomials for n=4
is used in the above scheme ,namely

1	-3	3
	2	-6
		6
4	20	36

Minimal values of h are given by :- 0=

$$2(1-:9335h+:4357h^2)(-:9335+:8714h)+2(1+:5624h+:1581h^2)(:5624+:3162h)$$
$$+2(1+1:2394h+:7680h^2)(1:2394+1:5361h)-\frac{(3+:8683h+1:3618h^2)}{2}(:8683+2:72h)$$
$$-\frac{(3+5:2141h+2:0264h^2)}{10}(5:2141+4:0528h)-\frac{(3+4:8315h+:5226h^2)}{18}(4:8315+1:0452h)$$

This reduces to:)- $:5128h^3-1:5596h^2+1:3757h-:3247=0=f(h)$ say

Using Newton's method of approximation to the root of an equation ,try h=0:4. A better solution is $h=0:4+h'$

where $h'=f(0:4)/f'(0:4)=-:008863/;3742=-:0236$

i.e. h=:3764. A better solution still is $h=:3764+:0005496/:4196$
$=:3777$

and the residual for this value of h is

$$:7096^2+1:2350^2+1:5777^2-3\frac{:5223^2}{4}-5\frac{:2585^2}{20}-1\frac{:1006^2}{36}=:00001$$

Thus we shall expect e^{hR} to lead to a good approximation.

R	$e^{:3777R}$	$\hat{\Sigma}_1$	$\hat{\Sigma}_1^2$	$\hat{\Sigma}_1^3$	$(e^{:3777R})^2$
-	000000	0	0	0	
-:9335	:7029	:7029	:7029	:7029	0
0:5624	1:2366	1:9395	(1:67265)		:494,068
1:2394	1:5969	5:4567	(8:64765)		1:529,180
1:5567	1:8003	3:8598	5:9193	7:9788	2:550,090
1:9129	2:0595	2:0595	2:0595	2:0595	3:241,080
					4:241,540
					12:055,958

$m_{\{0\}}=7:3962$ $m_{\{1\}}'=6:9750$ $m_{\{2\}}=8:6817$

Using the table of Tchebychef polynomials on page 48 we find the values of the a's thus:-

		$a_i S\ RT_i(x)$	$S^2/(n-1-i)$
$a_0=\frac{7:3962}{6}$	$=1:2327$	9:117296	2:938,662/5
$a_1=\frac{13:95}{70}$	$=0:199286$	2:780040	0:158,622/4
$a_2=-\frac{7:0794}{336}$	$=-0:0210696$	0:149160	0:009,462/3

The value of S^2 after a_2 namely :009,462 is sufficiently small to allow a good fit. Using the quadratic expression ,we have

$\mu V_0=1:401,257$ $\quad \& W_0=\ :398,572$ $\quad \mu\&^2V_0\ =-:126,417,6$

TWO HALF-TABLES OF DIFFERENCES FOR V AND W

V	&V	$\&^2V$	W	&W
1:401,257		-:126,418	0:199,286	
	-:126,418			:398,572
1:274,839		-:126,418	0:597,858	
	-:252,836			:398,572
1:022,003		-:126,418	0:996,430	
	-:379,254			:398,572
0:642,749			1:395,002	

Combining corresponding values of V and W we have

$e^{:3777R}$ computed	R	1:03 x Decimal	Frequency
:0256	$-\infty$	0	
:6770	-1:0328	1554	39
1:2020	0:4871	7075	139
1:6005	1:2452	9203	53*
1:8727	1:6611	9801	15
2:0184	1:8593	9976	4
2:0378	1:8847	9993	1
			251

* this 53 is actually 53:56 but it is reduced to 53 to make the total frequency computed the same as the observed total.

The curve is limited in both directions , extending from $x=0$ to $x=5:648$. It is necessary to introduce the factor 1:03 in order to make

$$\int_{-\infty}^{\infty} y\,dx = \int_{0}^{5:648} y\,dx = 1.$$

In effect we assume that the curve is $y = A\,R'(x)\,e^{-R^2(x)}$ where A is approximately unity. After the determination of $R(x)$ with $A=1$ we find the true value of A.

Comparison with Pearson's Type 111 Curve.

Data	44	135	45	12	8	3	1	3
Kapteyn	39	139	53	15	4	1		
Pearson	59	111	45	20	9	4	2	1

Grouping the frequencies 12,8,3,1,3 we compute the value of χ^2 .

(a) Kapteyn	(b) Pearson
$\chi^2 = 4:42$	$\chi^2 = 11:25$
$n = 4-3=1$	$n=4-2=2$
$P_{\chi^2} = :04$	$P_{\chi^2} < :01$

Clearly the Kapteyn graduation is superior to Pearson's Type 111 but P_{χ^2} for the Kapteyn curve is not conclusive being near the limit :05.

The equation of the transformation is

$$e^{:3777R(x)} = :025570 + :714622x - :06321x^2$$

EQUATION OF THE CURVE

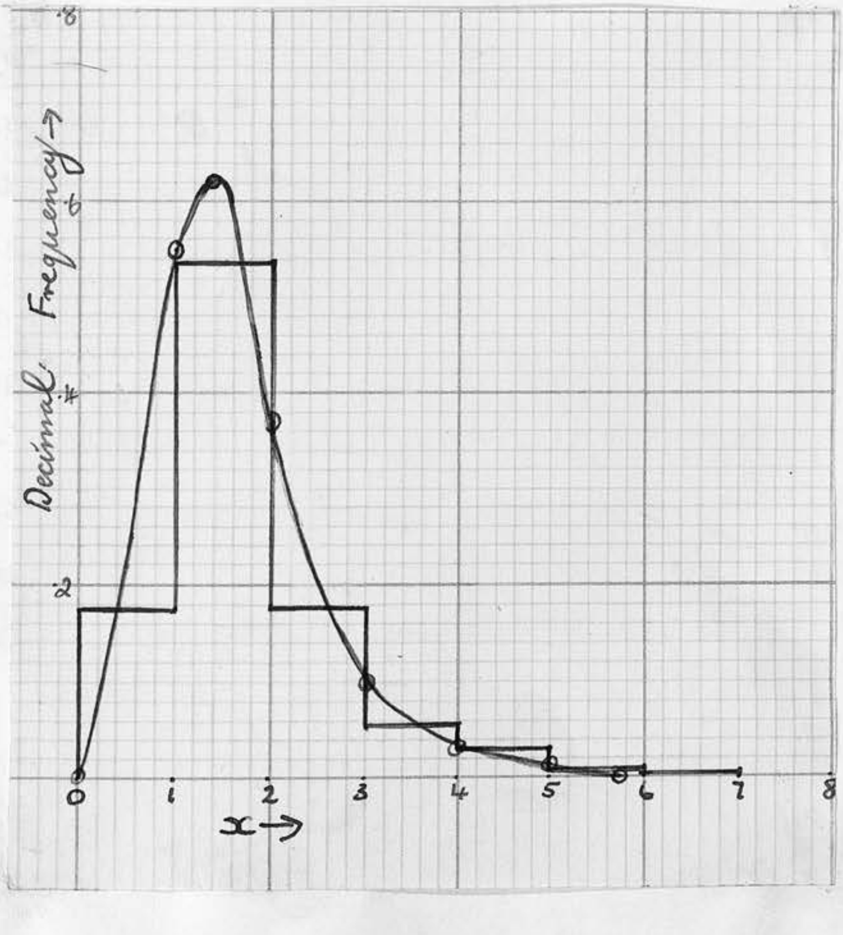
This is found to be $Y=$

$$\frac{1{:}03(:714622-:12642x)\ (:3777)^{-1}}{\sqrt{2\,\Pi}\,(:02557+:714622x-:06321x^2)}\ e^{-\frac{1}{2}\left(\frac{\log(:02557+:714622x-:06321x^2)}{:3777}\right)^2}$$

ORDINATES	$x=$	$Y=$
	0	0
	1	:5544
	2	:3713
	3	:1050
	4	:0305
	5	:0079

The maximum ordinate occurs at x=1:26 and =0:62 approx. .

"NUMBER OF WIVES " data.



Example 4.

Data "Valuation of House Property in England and Wales 1885-1886" Pearson, Phil. Trans Vol 186 page 396 and Kapteyn "Skew Frequency Curves" 1903 p.35.

x	Number of houses in 1000's.	Decimal	First Sums	
under£10	3175	5446	5446	
10-20	1451	2489	7935	
20-30	441:6	0758	8693	
30-40	259:8	0446	9139	
40-50	151:0	0259	9398	
50-60	90:4	0155	9553	
60-80	104:1	0179	9641	(60-70 group)
80-100	47:3	0081		
100-150	58:9	0101		
150-300	38:0	0065		
300-500	8:8	0015		
500-1000	3:0	0005		
1000-1500	1:0	0002		
	5829 :9			

(N.B. Throughout this example Kapteyn's table of $\frac{1}{\sqrt{\pi}}\int_{-\infty}^{R} e^{-t^2}dt$ is used)

This distribution is what is usually called "J" shaped and is distinct from the other examples .

We take the five data,

0 :5446 :7935 :8693 :9139

and determine h of the transformation

$hR=\log_e(a_0 + a_1T_1(x) + a_2T_2(x) + a_3T_3(x) + \ldots\ldots)$

After finding this h we proceed to graduate the data using 96% of the total frequency to determine the unknowns.

As before e^{hR} or $1 + hR + \frac{h^2R^2}{2!}$ is set out and the moments calculated. We assume that mod. hR is less than 1.

Freq.	R	e^{hR}
0	$-\infty$	0
:5446	:0793	$1+{:}0793h + {:}0063h^2/2$
:7935	:5788	$1+{:}5788h + {:}3350h^2/2$
:8693	:7940	$1+{:}7940h + {:}6304h^2/2$
:9139	:9655	$1+{:}9655h + {:}9322h^2/2$

The first three factorial moments of e^{hR} are

$$m_0 = 4 + 2{:}4176h + 1{:}9039h^2/2$$

$$m_1 = 10 + 7{:}4809h + 6{:}2963h^2/2$$

$$m_2 = 10 + 8{:}7538h + 7{:}8194h^2/2$$

Using the Tchebychef table for n=5, namely

$$\begin{array}{lcccc}
 & 1 & -4 & 6 & -4 \\
 & & 2 & -9 & 12 \\
 & & & 6 & -20 \\
 & & & & 20 \\
\hline
ST_r^2(x) & 5 & 40 & 126 & 160
\end{array}$$

we have $a_0 = (4+2{:}4176h + {:}9519h^2)/5$

$$a_1 = (4+5{:}2914h+2{:}4885h^2)/40$$

$$a_2 = (-6-{:}2997h + {:}8365h^2)/126$$

$$a_3 = (4-1{:}8556h - {:}6320h^2)/160$$

The residual after a_3 is $S^2 = (1+{:}0793h+{:}0032h^2)^2+(1+{:}5788h+{:}1675h^2)^2$

$$+(1+{:}7940h+{:}3152h^2)^2+(1+{:}9655h+{:}4661h^2)^2-\frac{(4+2{:}4176h+{:}9519h^2)^2}{5}$$

$$-\frac{(4+5{:}2914h+2{:}4885h^2)^2}{40}-\frac{(6+{:}2997h-{:}8365h^2)^2}{126}-\frac{(4-1{:}8556h-{:}6320h^2)^2}{160}$$

The values of h which make this a minimum are roots of $\frac{ds^2}{dh}=0$

$$=(1+{:}0793h+{:}0032h^2)({:}0793+{:}0063h)+(1+{:}5788h+{:}1675h^2)({:}5788+{:}335h)$$

$$+(1+{:}7940h+{:}3152h^2)({:}7940+{:}6304h)+(1+{:}9655h+{:}4661h^2)({:}9655+{:}9322h)$$

$$-\frac{(4+2{:}4176h+{:}9519h^2)(2{:}4176+1{:}9038h)}{5}-\frac{(4+5{:}2914h+2{:}4885\ h^2)(5{:}2914+4{:}977h)}{40}$$

$$-\frac{(6+{:}2997h-{:}8365h^2)({:}2997-1{:}673h)}{126}+\frac{(4-1{:}8556h-{:}632h^2)(1{:}8556+1{:}264h)}{160}$$

This is a cubic equation in h and simplifies to

$${:}00116h^3 + {:}00812h^2 + {:}00722h - {:}01360 = 0 = f(h) \text{ say.}$$

$$\frac{df(h)}{dh} = f'(h) = {:}00348h^2 + {:}01624h + {:}00722$$

As a solution of this equation try h=0:8 .Then a better approximation is $h=0{:}8 + \frac{:002}{:0224} = 0{:}8893$ by using Newton's approximation method.

Again f(:8893)=:00005829 f'(:8893)=:024414

and finally $h= 0{:}8893 - \frac{:00005829}{:024414} = 0{:}8869$

Actually f(0:8869) =0:000,000,2 and so we have a good approximation to the root.

Using this value of h we graduate $e^{hR} = e^{:8869R}$

R	$e^{:8869R}$	$\hat{\Sigma}_1$	$\mu\hat{\Sigma}_1^2$	$\hat{\Sigma}_1^3$	$\mu\hat{\Sigma}_1^4$	$\hat{\Sigma}_1^5$	$(e^{hR})^2$
-	0	0	0	0	0	0	0
0:0793	1:0728	1:0728	1:0728	1:0728	1:0728	1:0728	1:150,900
0:5788	1:6708	2:7436	3:8164	4:8892	(3:5174)		2:791,573
0:7940	2:0222	4:7658	(6:1993)				4:089,293
0:9655	2:3544	10:9982	(23:2326)				5:543,199
1:0982	2:6485	8:6438	17:7335	29:9176	(30:2373)		7:014,552
1:2008	2:9009	5:9953	9:0897	12:1841	15:2785	18:3729	8:415,221
1:2736	3:0944	3:0944	3:0944	3:0944	3:0944	3:0944	9:575,311
							38:580,049

The Central Moments are:-

$m_{\{0\}}$ =15:7640 $m'_{\{1\}}$=17:0333 $m_{\{2\}}$ = 34:8068 $m'_{\{3\}}$=26:7199

$m_{\{4\}}$ = 19:4457

CENTRAL TABLE FOR n=8

m_r \ $a_r \rightarrow$	1:970500	:202777	-:018267	:003545	-:001158
15:7640	+1		-15		+45
17:0333		+2		-30	
34:8068			+6		-60
26:7199				+20	
19:4457					+70
S $T_r^2(x)$	8	168	1512	6600	15400

RESIDUAL TABLE

	$a_i S\ RT_i(x)$	S^2	$S^2/(n-1-i)$
$a_0 = \frac{15:7640}{8} = 1:9705$	31:062,962	7:517,087	1:07
$a_1 = \frac{34:0666}{168} =:202777$	6:907,923	0:609164	0:10
$a_2 = \frac{-27:6192}{1512} =-:018267$	0:504,520	0:104644	0:02
$a_3 = \frac{23:3990}{6600} =:003545$	0:082,949	0:021695	0:005
$a_4 = \frac{-17:8290}{15400} =+001158$	0:020,646	0:001049	0:0003

The quartic transformation is chosen and we have :-

$\mu V_0 = 2:192,395$ $\delta W_0 =:299,204$ $\mu\delta^2 V_0 =-:040,122$

$\delta^3 W_0 =:070,900$ $\mu\delta^4 V_0 =-:081,060$

Build up two half tables of differences for Vand W and obtain:-

e^{hR}	e^{hR} Observed	$(e_A^{hR}-e_C^{hR})^2$	hR Computed	R
0:0028	0	:000,008		$-\infty$
1:0657	1:0728	:000,050	:063594	:0717
1:6680	1:6708	:000,008	:511620	:5769
2:0428	2:0222	:000,424	:714322	:8054
2:3420	2:3544	:000,154	:851000	:9595
2:6365	2:6485	:000,144	:969439	1:0931
2:9162	2:9009	:000,234	1:070258	1:2067
3:0899	3:0944	:000,020	1:128175	1:2720
		:001,042		

The check column $S\ (e_A^{hR} - e_C^{hR})^2$,giving the sum of the squares of the differences is in good agreement with S^2=:001049

Computed Decimal

0	:5403	:7927	:8727	:9125	:9389	:9561	:9640

1:037 Decimal Difference

0	:5603	:2617	:0830	:0413	:0273	:0179	:0082

Frequency

0-10	10-20	20-30	30-40	40-50	50-60	60-70
3266	1526	484	241	159	104	48

Comparison of the Kapteyn Curve with Pearson's .

Observed		Kapteyn		Pearson.
0-10	3175	3266		3602
10-20	1451	1526		1051
20-30	442	484		455
30-40	260	241		255
40-50	151	159		154
50-60	90	104		98
60-80	104	48	(60-70)	103
80-100	47			46
100-1500	110			84

The Kapteyn curve lies closer to the observations for Pearson's is 427 in excess in the 3175 frequency. There is one defect in the Kapteyn curve and that is that the frequencies in the 60-80,80-100 ...... groups are eliminated.These frequencies constitute 3:5% of the total frequency and so the defect is not serious.Further the distribution is "J" shaped but the Kapteyn curve is not.

EQUATION OF THE CURVE AND THE VALUE OF ORDINATES.

The transformation is :-

$$e^{:8869R(X)} = 2:192395 + :149602(2X) - :020061(X^2-\tfrac{1}{4}) + :005908(2X)(X^2-\tfrac{1}{4}) - :0033775(X^2-\tfrac{1}{4})(X^2-9/4)$$

and $$:8869R'(x)\, e^{:8869R(x)} = :29624983 - :01161725(2x) + :00886249(2x)^2 - :00168875(2x)^3$$

The equation of the curve is thus $y=$

$$1{:}423376\Big({:}2962498-{:}011617(2x)+{:}008862(2x)^{2}-{:}001689(2x)^{3}\Big)$$

$$\mathrm{X}\ \frac{1}{\sqrt{\Pi}}\ \mathrm{Exp.}-\Big({:}44345+\log\Big\{2{:}192395+{:}149602(2x)-{:}020061(x^{2}-\tfrac{1}{4})+ {:}005908(2x)(x^{2}-\tfrac{1}{4})-{:}003377(x^{2}-\tfrac{1}{4})(x^{2}-\tfrac{9}{4})\Big\}/{:}8869\Big)^{2}$$

$$=E(x)\ \frac{1}{\sqrt{\Pi}}\ \mathrm{Exp.}\ f(x) \quad \text{say.}$$

VALUE OF ORDINATES.

x	$E(x)$	$1/\sqrt{\Pi}$ Exp. $f(x)$	y
-3:5	-	0	0
-2:5	1:1202	:4327	:4847
-1:5	:6497	:1993	:1295
-0:5	:4532	:1185	:0537
+0:5	:4153	:0788	:0327
+1:5	:4207	:0532	:0224
+2:5	:3540	:0371	:0131
+3:5	:0996	:0298	:0030

The equation of the curve can be written

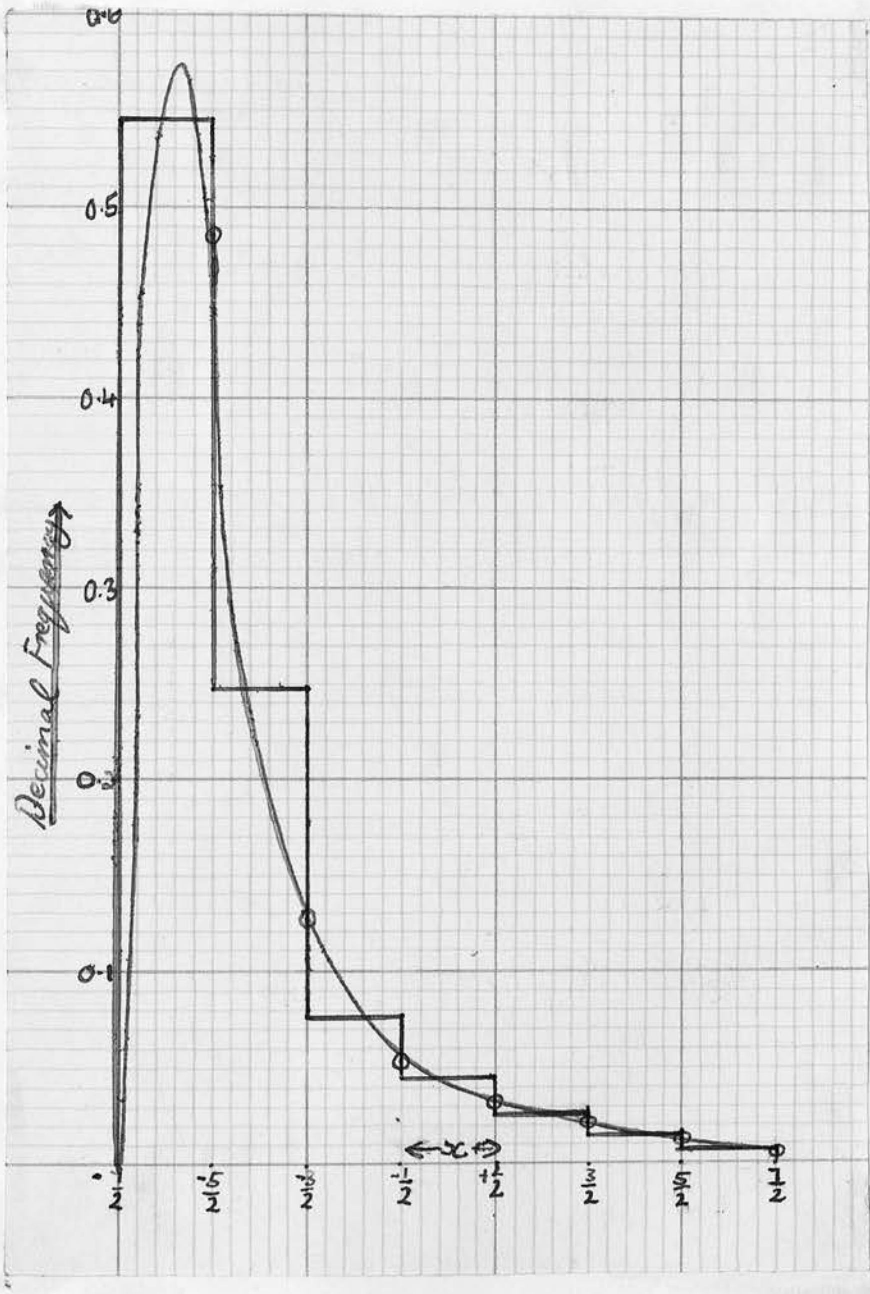
$$y=\frac{1{:}037}{\sqrt{\Pi}}\quad R'(x)\ e^{-R^{2}(x)}$$

where $${:}8869R(x)=\mathrm{LOG}_{e}\Big(2{:}192395+{:}149602(2x)-{:}020061(x^{2}-\tfrac{1}{4}) +{:}005908(2x)(x^{2}-\tfrac{1}{4})-{:}003377(x^{2}-\tfrac{1}{4})(x^{2}-\tfrac{9}{4})\Big)$$

The graph of the curve is given overleaf.

"VALUATION OF HOUSE PROPERTY " Data

KAPTEYN LOGARITHMIC CURVE FITTED



CHAPTER TWO

PROPERTIES OF ORTHOGONAL POLYNOMIALS RELATED TO THEIR GENERATING FUNCTIONS.

(a) Theorems concerning frequency generating functions and Orthogonal Polynomials.

(b) Orthogonal Polynomials defined by

1/ $H_r(x)\ f(x) = \Delta^r\ x^{(r)}\ f(x)$

2/ $H_r(x)\ g(x) = \Delta^r\ x^{(r)}\ f(x)$

(c) Relations between Orthogonals Polynomials and their basic functions.

(d) The generating function of $f(x)\ g(\Delta)\ x^{(r)}$.

(e) A new approach to Gram Polynomials.

(a) Theorems concerning frequency generating functions and orthogonal polynomials.

By definition if $f(x)$ is generated by $F(t)$ then

$$F(t) = \overset{n}{\underset{0}{S}} f(x)\, t^{x}$$

in the case of the discrete variable . Again if $F(a)$ is the factorial moment generating function of $f(x)$ then $a^r F(a)$ generates $(-)^r \Delta^r f(x-r)$ where $t=1+a$.

In just the same way $a^r F(t) = S\, t^x f(x)\, (t-1)^r$

$$=(-)^r S\, t^x \left[f(x) - r f(x-1) + r_{(2)} f(x-2) - \ldots\ldots\ldots\ldots\ldots \right]$$

$$=(-)^r S\, t^x \Delta^r f(x-r)$$

Clearly $\dfrac{d^s F(t)}{dt^s} = S\, x^{(s)} t^{x-s} f(x)$ where $x^{(s)} = (x)(x-1)(x-2)\ldots(x-s+1)$

and so $t^s \dfrac{d^s}{dt^s} F(t) = S\, x^{(s)} t^x f(x)$

i.e. $t^s \dfrac{d^s}{dt^s} F(t)$ generates $x^{(s)} f(x)$ .... (1)

Similarly $a^s \dfrac{d^s}{dt^s} F(t)$ generates $(-)^s \Delta^s x^{(s)} f(x)$ .... (2)

Theorem 1

If $H_r(x) = a_0 + a_1 x + a_2 x^{(2)} + a_3 x^{(3)} + \ldots\ldots\ldots\ldots\ldots + a_r x^{(r)}$

where $a_0, a_1, a_2, \quad a_r$ are constants ,

then $H_r\left(t \dfrac{d}{dt} \right) F(t)$ generates $H_r(x) f(x)$

where

$$F(t) = S\, t^x f(x)$$

Stated in full, $H_r(t\,D)\, F(t) = S\, t^x H_r(x)\, f(x)$

where $D = \dfrac{d}{dt}$

To prove this it is to be observed that

$$(tD)^{(s)}=tD(tD-1)(td-2).......(tD-s+1) =t^s \frac{d^s}{dt^s}$$

For example

$$(tD)^{(2)}= tD(tD-1) =t(D+tD^2) -tD =t^2D^2$$

Similarly $(tD)^{(3)}= t^3D^3$

Hence assume the truth of the above up to r say ,i.e.

$$(tD)^{(r)} = t^r D^r$$

Then

$$(tD)^{(r+1)}= (tD)^{(r)}(tD-r)$$

$$= tD\, t^rD^r -rt^rD^r$$

$$=t(rt^{r-1}D^r +t^rD^{r+1}) -rt^rD^r$$

$$= t^{r+1} D^{r+1}$$

which is the form $(tD)^{(r)}$ assumes if r+1 is substituted for r.

Hence applying the operator $(tD)^{(r)}$ to any polynomial form, we can assert that $(tD)^{(r)} = t^rD^r$

Now

$$H_r(tD)\, F(t)=\left[a_0+a_1tD +a_2(tD)^{(2)}+ a_3(tD)^{(3)}+.....\right]F(t)$$

$$=\left[a_0+ a_1tD +a_2t^2D^2 + a_3t^3D^3+......\right]F(t)$$

$$=a_0F(t)+a_1tF'(t) +a_2t^2F''(t) +.....$$

which by (1) generates

$$\left[a_0 +a_1x + a_2x^{(2)}+ a_3 x^{(3)}+ \right]f(x)$$

$$=H_r(x)f(x) \quad \text{since } F(t) = S\, t^xf(x)$$

Hence $H_r(tD)\, F(t)$ generates $H_r(x)f(x)$

Further Theorems.

(a) Assume $G(t)$ can be expanded as a Taylor Series,

i.e. $G(t) = a_o + a_1 t + a_2 t^2 + \dots\dots\dots\dots$

then $G(a/t) = a_o + a_1 \frac{a}{t} + a_2 \frac{a^2}{t^2} + a_3 \frac{a^3}{t^3} + \dots\dots$

and

$G(-a/t)\,F(t) = a_o F(t) + a_1(-a/t)F(t) + a_2(-a/t)^2 F(t) + \dots\dots\dots$

Now $a^s t^{-s} F(t)$ generates $(-)^s \Delta^s f(x)$

Hence

$G(-a/t)\,F(t)$ generates

$a_o f(x) + a_1 \Delta f(x) + a_2 \Delta^2 f(x) + \dots\dots\dots\dots$

$= \underline{G(\Delta)\,f(x)}$ $\dots\dots\dots\dots\dots$ (3)

(b) Again $G(t)\,W(t) = a_o W(t) + a_1 t W(t) + a_2 t^2 W(t) + \dots\dots$

generates

$a_o w(x) + a_1 w(x-1) + a_2 w(x-2) + \dots\dots\dots\dots$

where $W(t) = S\ t^x\,w(x)$

But $w(x-s) = (1-\nabla)^s\,w(x)$ where $\nabla u_x = u_x - u_{x-1}$

and so $G(t)\,W(t)$ generates $G(1-\nabla)\,w(x)$ $\dots\dots$ (4)

(c) If we can expand $G(1/t)$ in powers of $1/t$, then

$G(1/t)\,W(t) = a_o W(t) + a_1 t^{-1} W(t) + a_2 t^{-2} W(t) + \dots\dots\dots\dots$

generates $a_o w(x) + a_1 w(x+1) + a_2 w(x+2) + \dots\dots\dots\dots$

But $w(x+s) = (1+\Delta)^s w(x)$

Hence $G(1/t)\,W(t)$ generates $G(1+\Delta)\,w(x)$ $\dots\dots\dots$ (5)

(b) Orthogonal Polynomials defined by
(1) $H_r(x)f(x) = \Delta^r x^{(r)}f(x)$ (2) $H_r(x)f(x) = \Delta^r x^{(r)} w(x)$

(1) If $H_r(x)\, f(x) = \Delta^r x^{(r)} f(x)$, what can we say about $H_r(x)$? Under what conditions is it defined? We shall attempt an answer to these questions.

Assume $H_r(x) = a_0 + a_1 x + a_2 x^{(2)} + \ldots\ldots + a_r x^{(r)}$.
where $a_0, a_1, \ldots \ldots a_r$ are constants. $H_r(x)$ is thus a polynomial in x of degree r expressed as factorials in x .

Now if $H_r(x)f(x) = \Delta^r x^{(r)} f(x)$ then

$$S\ t^x H_r(x)\, f(x) = S\ t^x \Delta^r x^{(r)} f(x)$$

But by theorem 1, $H_r(\, tD)\, F(t)$ generates $H_r(x)\, f(x)$

$$\text{Hence}\quad S\ t^x H_r(x)\, f(x) = H_r(tD)\, F(t) = S\ t^x \Delta^r x^{(r)} f(x)$$

$$= (-)^r a^r \frac{d^r}{dt^r} F(t) \quad \text{by (2)}$$

$$\text{i.e.}\quad H_r\left(t \frac{d}{dt}\right) F(t) = (1-t)^r \frac{d^r}{dt^r} F(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\underline{6})$$

In other words if $F(t)$ is given and its first r derivatives exist and are continuous for a given range of t, then orthogonal polynomials with respect to $f(x)$ exist if $H_r(x)$ is uniquely determined by equation (6) , provided these polynomials are of the form

$$H_r(x)\, f(x) = \Delta^r x^{(r)} f(x)$$

In (6) write $H_r(\, tD) = a_0 + a_1 tD + a_2 t^2 D^2 + \ldots\ldots\ldots$

which we have shown is the same as

$$a_0 + a_1(tD)^{(1)} + a_2 (tD)^{(2)} + a_3(tD)^{(3)} + \dots + a_r(tD)^{(r)}$$

where $D = \dfrac{d}{dt}$

Then

$$a_0F(t) + a_1tF'(t) + a_2t^2F''(t) + a_3t^3F'''(t) + \dots + a_rt^rF^r(t)$$

$$= (1-t)^r F^r(t)$$

where we have written $F^s(t)$ for $\dfrac{d^s}{dt^s} F(t)$.

Now assume $F(t)$ and its first $2r$ derivatives exist and are continuous . Expand $F(t)$, $F'(t)$, $F''(t)$, etc. $F^r(t)$ in powers of t and equate coefficients in equation (6).

Then:-

$$a_0\frac{F(0)}{0!} \quad \dots \dots \dots \dots \quad -\frac{F^r(0)}{0!} = 0$$

$$a_0\frac{F'(0)}{1!} + a_1\frac{F'(0)}{0!} \dots \dots \dots \dots -\left[\frac{F^{r+1}(0)}{1!} - rF\frac{^r(0)}{0!}\right] = 0$$

$$a_0\frac{F''(0)}{2!} + a_1\frac{F''(0)}{1!} + a_2\frac{F''(0)}{0!} \dots \dots \dots -\left[F^{r+2}\frac{(0)}{2!} - rF^{r+1}\frac{(0)}{1!} + r_{(2)}\frac{F^r(0)}{0!}\right] = 0$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \dots\dots\dots\dots$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \dots\dots\dots\dots$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \dots\dots\dots\dots$$

$$a_0\frac{F^r(0)}{r!} + a_1\frac{F^r(0)}{(r-1)!} + a_2\frac{F^r(0)}{(r-2)!} \dots \dots \dots -\left[\frac{F^{2r}(0)}{r!} - r\frac{F^{2r-1}(0)}{(r-1)!} + r_{(2)}\frac{F^{2r-2}(0)}{(r-2)!} \quad \text{------} \right] = 0$$

These equations determine $a_0, a_1, a_2 \dots\dots\dots a_r$ when $F(t)$ is given, provided

$$\begin{vmatrix} \frac{F(0)}{0!} & & & & & \\ \frac{F'(0)}{1!} & \frac{F'(0)}{0!} & & & & \\ \frac{F''(0)}{2!} & \frac{F''(0)}{1!} & \frac{F''(0)}{0!} & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \frac{F^r(0)}{r!} & \frac{F^r(0)}{(r-1)!} & \frac{F^r(0)}{(r-2)!} & & & \frac{F^r(0)}{0!} \end{vmatrix} \neq 0$$

which is certainly true is $F(0), F'(0), F''(0), \dots . F^r(0) \neq 0$.

With this assumption the form of $H_r(x)$, by well known results in determinants , is:-

$$(7)\dots\dots\dots \quad H_r(x)=(-)^r \begin{vmatrix} 1 & x & x^{(2)} & x^{(3)} & & x^{(r)} & 0 \\ \frac{1}{0!} & \cdot & \cdot & \cdot & & \cdot & -\frac{F^r(0)}{F(0)} \\ \frac{1}{1!} & \frac{1}{0!} & \cdot & \cdot & & \cdot & -\frac{\left[F^{r+1}(0) - rF^r(0)\right]}{F'(0)} \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & \cdot & & \cdot & -\frac{\left[\frac{F^{r+2}(0)}{2!} - r\frac{F^{r+1}(0)}{1!} + r_{(2)}\frac{F^r(0)}{0!}\right]}{F''(0)} \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & & \cdot & -\frac{\left[\frac{F^{r+3}(0)}{3!} - r\frac{F^{r+2}(0)}{2!} + r_{(2)}\frac{F^{r+1}(0)}{1!} - r_{(3)}\frac{F^r(0)}{0!}\right]}{F'''(0)} \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \dots\dots\dots\dots\dots\dots \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \dots\dots\dots & & 1 & -\frac{\left[\frac{F^{2r}(0)}{r!} - r\frac{F^{2r-1}(0)}{(r-1)!} + r_{(2)}\frac{F^{2r-2}(0)}{(r-2)!} - \dots\right]}{F^r(0)} \end{vmatrix}$$

Hence we can say that if $F(t)$ is such that its first r derivatives exist and do not vanish for t=0, and further its first 2r derivatives exist and are continuous for a specified range of t, then the polynomials of degree r defined by the determinant (7) above satisfies the equation

$H_r(x)f(x) = \Delta^r x^{(r)} f(x)$ where $F(t)$ generates $f(x)$.

The polynomials $H_r(x)$ are moreover orthogonal with respect to $f(x)$. For $(1-t)^r F^r(t) = \mathop{S}_0^n t^x \Delta^r x^{(r)} f(x) = \mathop{S}_0^n t^x H_r(x) f(x)$

If n is finite we can differentiate both sides with respect to t and obtain

$$\left[\frac{d^s}{dt^s}(1-t)^r F^r(t)\right]_{t=1} = \mathop{S}_0^n x^{(s)} H_r(x) f(x)$$

$$= 0 \qquad s \text{ less than } r$$

$$= (-)^r r!\, F^r(1) \qquad s=r$$

provided $F(t)$ and its first 2r derivatives are bounded for t=1.

Clearly $\left[\frac{d^s}{dt^s} H_r\left(t\frac{d}{dt}\right)F(t)\right]_{t=1} = 0 \qquad s$ less than r

and under the conditions already stated this leads to

$$H_r(x) = \begin{vmatrix} 1 & x & x^{(2)} & \cdots & x^{(r)} \\ F(1) & F'(1) & F''(1) & \cdots & F^r(1) \\ F'(1) & F''(1)+F'(1) & F'''(1)+2F''(1) & \cdots & F^{r+1}(1)+rF^r(1) \\ \vdots & \vdots & & & \vdots \\ F^{r-1}(1) & F^r(1)+\overline{r-1}F^{r-1}(1) & & \cdots & F^{2r-1}(1)+r^{(2)}F^{2r-2}(1)+\cdots \end{vmatrix} \qquad (8)\ldots\ldots$$

When it is remembered that $F(t) = \sum_{0}^{n} t^x f(x)$ and so

$$F^s(1) = \sum x^{(s)} f(x)$$

$$= \text{sth factorial moment of } f(x)$$

the form of $H_r(x)$ in (8) could be expressed in terms of the factorial moments of $f(x)$ where the (r,s)th. term (excluding the first row of the determinant) is

$$p_{r.s} = F^{r+s}(1) + r.sF^{r+s-1}(1) + r_{(2)}s^{(2)}F^{r+s-2}(1) + \ldots\ldots\ldots$$

$$= m_{(r+s)} + r.s.m_{(r+s-1)} + r_{(2)}s^{(2)}m_{(r+s-2)} + \ldots\ldots$$

where $m_{(k)}$ denotes the k th factorial moment of $f(x)$. The factorial moment form of the determinant for $H_r(x)$ arises otherwise from combining $H_r(x) = a_0 + a_1x + a_2x^{(2)} + \ldots a_r x^{(r)}$ with

$$\sum_{0}^{n} x^{(s)} H_r(x) f(x) = 0 \quad \text{for all s less than r}$$

and eliminating the unknowns $a_0, a_1, a_2, \ldots\ldots a_r$.

(See the Determinantal Method of determining Tchebychef's polynomials used by A.C.Aitken in "Data by the Orthogonal Polynomials of Least Squares" ,ref.2 . Also Gonin ,H.T. "On the Orthogonal Polynomials in Binomial and Hypergeometric Distributions" ref. 16.)

(11) The more general type of polynomial equation

$$H_r(x)\, w(x) = (-)^r \Delta^r x^{(r)} f(x) \quad \text{has a}$$

solution, as we should expect, similar to the above. For using the same reasoning as in (1) ,this equation implies

$$H_r(tD)W(t) = (t-1)^r F^r(t) \qquad \text{where } W(t)=\sum_0^n t^x w(x)$$

and provided $F(0), F'(0), F''(0), \ldots\ldots\ldots\ldots F^{2r}(0)$ exist and are finite, and $W(0), W'(0), W''(0), \ldots\ldots\ldots W^r(0)$ do not vanish then

$$H_r(x) = (-)^r \begin{vmatrix} 1 & x & x^{(2)} & \ldots\ldots & x^{(r)} & 0 \\ \frac{1}{0!} & \cdot & \cdot & & & F^r(0)/W(0) \\ \frac{1}{1!} & \frac{1}{0!} & \cdot & & & \left(\frac{F^{r+1}}{1!} - r\frac{F^r}{0!}\right)/W'(0) \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & & & \left(\frac{F^{r+2}}{2!} - r\frac{F^{r+1}}{1!} + r_{(2)}\frac{F^r}{0!}\right)/W''(0) \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & & & \left(\frac{F^{r+3}}{3!} - r\frac{F^{r+2}}{2!} + r_{(2)}\frac{F^{r+1}}{1!} - r_{(3)}\frac{F^r}{0!}\right)/W'''(0) \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & & \frac{1}{0!} & \left(\frac{F^{2r}}{r!} - r\frac{F^{2r-1}}{(r-1)!} + r_{(2)}\frac{F^{2r-2}}{(r-2)!} \ldots\right)/W^r(0) \end{vmatrix} \qquad (9)$$

where the $(r+2)$ th. column has as its s th. term

$$p_{s,r+2} = \left(\frac{F^{r+s}}{s!} - r\frac{F^{r+s-1}}{(s-1)!} + r_{(2)}\frac{F^{r+s-2}}{(s-2)!} \ldots\ldots + (-)^s r_{(s)}\frac{F^r}{0!} \right) / W^s(0)$$

where we have written F^a for $\frac{d^a}{dt^a} F(t)$ when $t=0$.

We turn now to several applications.

Example 1.

Take $F(t) = e^{m(t-1)}$ and so $f(x) = e^{-m}\frac{m^x}{x!}$

and $F(t)$ is the frequency generating function of the Poisson distribution.

Now $\Delta^r x^{(r)} f(x) = \Delta^r x^{(r)} \dfrac{e^{-m} m^x}{x!} = \Delta^r \dfrac{e^{-m} m^x}{(x-r)!}$

$= \dfrac{e^{-m} m^x}{x!} H_r(x)$ where $H_r(x)$ is some polynomial in x of degree r.

But since $\Delta^r x^{(r)} f(x) = H_r(x)\, f(x)$ the form of $H_r(x)$ is that given on page 68 (7) . Using $F^s(0) = m^s F(0)$,we have

$$H_r(x) = (-)^r \begin{vmatrix} 1 & x & x^{(2)} & \cdots\cdots & x^{(r)} & 0 \\ \frac{1}{0!} & \cdot & \cdot & & \cdot & -m^r \\ \frac{1}{1!} & \frac{1}{0!} & \cdot & & \cdot & -(m-r)m^{r-1} \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & & \cdot & -(m^2/2! - rm + r_{(2)})m^{r-2} \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & \cdot & -(\frac{m^3}{3!} - \frac{rm^2}{2!} + r_{(2)} m - r_{(3)})m^{r-3} \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \cdots\cdots & & \frac{1}{0!} & -(\frac{m^r}{r!} - \frac{rm^{r-1}}{(r-1)!} + r_{(2)}\frac{m^{r-2}}{(r-2)!} - \cdots) \end{vmatrix}$$

which reduces, by continued subtraction of rows, to

$$H_r(x) m^{-r} (-)^{r+1} = \begin{vmatrix} 1 & x & x^{(2)} & x^{(3)} & \cdots & x^{(r)} & 0 \\ 1 & \cdot & \cdot & \cdot & \cdots & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdots & \cdot & -r/m \\ \cdot & \cdot & 1 & \cdot & \cdots & \cdot & +r_{(2)}/m^2 \\ \cdot & \cdot & \cdot & 1 & \cdots & \cdot & -r_{(3)}/m^3 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & 1 & (-)^r/m^r \end{vmatrix}$$

and so

$H_r(x) = m^r(1 - rx/m + r_{(2)} x^{(2)}/m^2 \;\ldots\ldots\ldots\ldots)$

which is the r th polynomial of Charlier.

Moreover

$$\overset{\infty}{\underset{0}{S}} H_r(x)f(x)t^x = \overset{\infty}{\underset{0}{S}} t^x \Delta^r x^{(r)} f(x)$$

$$= \overset{\infty}{\underset{0}{S}} t^x \Delta^r \frac{e^{-m} m^{x-r}}{(x-r)!} m^r$$

$$= (-)^r m^r (t-1)^r e^{m(t-1)} \quad \text{by using } a^r G(t)$$

generates $(-)^r \Delta^r g(x-r)$ if $G(t)$ generates $g(x)$.

For clearly $\overset{n}{\underset{0}{S}} H_r(x)f(x)t^x = \overset{n}{\underset{0}{S}} t^x \Delta^r x^{(r)} f(x)$

$$= \overset{n}{\underset{0}{S}} t^x \Delta^r \frac{e^{-m} m^r}{(x-r)!} m^{x-r}$$

$$= (-)^r e^{-m} m^r (t-1)^r \overset{n}{\underset{0}{S}} \frac{t^x m^x}{x!}$$

Hence $\underset{n \to \infty}{\text{Limit}} \overset{n}{\underset{0}{S}} H_r(x)f(x)t^x = (-)^r m^r (t-1)^r e^{m(t-1)}$

Further $\overset{n}{\underset{0}{S}} H_r(x)f(x)\, x^{(s)} t^{x-s} = (-)^r e^{-m} m^r \frac{d^s}{dt^s} (t-1)^r \overset{n}{\underset{0}{S}} \frac{t^x m^x}{x!}$

Now $\overset{n}{\underset{0}{S}} \frac{t^x m^x}{x!}$ and its derivatives with respect to t have a finite limit as $n \to \infty$ for all finite values of m and t.

Hence $\overset{\infty}{\underset{0}{S}} H_r(x)\, f(x)\, x^{(s)} = (-)^r e^{-m} m^r \left[\frac{d^s}{dt^s} (t-1)^r e^{mt}\right]_{t=1}$

$$= 0 \quad \text{for s less than r}$$

$$= (-)^r e^{-m} m^r r!\, e^m \qquad s=r$$

i.e. $\overset{\infty}{\underset{0}{S}} \frac{H_r(x)}{m^r} f(x) \frac{x^{(r)}}{m^r} = (-)^r r!/m^r$

i.e. $\overset{\infty}{\underset{0}{S}} K_r(x)\, K_s(x)\, e^{-m} m^x / x! = 0 \qquad s \neq r$

$$= (-)^r r!/m^r \qquad r=s$$

where $K_r(x) = H_r(x) / m^r$ and is the rth polynomial in Frequencies of Type B. (See Ref. 11)

Example (11)

$$\text{Take } F(t) = (pt + q)^n \quad \text{for which}$$

$$f(x) = n_{(x)}\, p^x\, q^{n-x}$$

By examination of the first few differences of f(x) it becomes evident that $\Delta^r x^{(r)} f(x) = H_r(x)\, f(x)$

and so $\quad W(t) = F(t)$

and $\quad W^s(0) = F^s(0) = n^{(s)}\, p^s q^{n-s}$

and this is another example of case (1) page 66.

Hence $\quad H_r(x) =$

$$\begin{vmatrix} 1 & x & x^{(2)} & x^{(3)} & x^{(r)} & 0 \\ \frac{1}{0!} & \cdot & \cdot & \cdot & \cdot & n^{(r)} p^r q^{-r} \\ \frac{1}{1!} & \frac{1}{0!} & \cdot & \cdot & \cdot & \frac{n^{(r+1)} p^{r+1} q^{n-r-1} - rn^{(r)} p^r q^{n-r}}{npq^{n-1}} \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & \cdot & \cdot & \\ & & & & & \\ \frac{1}{r!} & \frac{1}{(r-1)!} & & & \frac{1}{0!} & n\frac{^{(2r)} p^{2r} q^{n-2r}}{r!} - rn\frac{^{(2r-1)} p^{2r-1} q^{2n-2r+1}}{(r-1)!} \end{vmatrix}$$

and by continued subtraction of rows ,i.e.

2nd row from 3rd, 4th, 5th ....
3rd row from 4th, 5th, 6th ....

dividing by factor in each case to obtain all zeros on L.H.S. except the diagonal elements ,we have

$$
(-)^r H_r(x) = \begin{vmatrix}
1 & x & x^{(2)} & x^{(3)} & & x^{(r)} & 0 \\
1 & \cdot & \cdot & \cdot & & \cdot & +n^{(r)} p^r q^{-r} \\
\cdot & 1 & \cdot & \cdot & & \cdot & -(n-1)^{(r-1)} p^{r-1} q^{-r} r \\
\cdot & \cdot & 1 & \cdot & & \cdot & +(n-2)^{(r-2)} p^{r-2} q^{-r} r_{(2)} \\
\cdot & \cdot & \cdot & 1 & & \cdot & -(n-3)^{(r-3)} p^{r-3} q^{-r} r_{(3)} \\
\cdot & \cdot & \cdot & \cdot & & \cdot & \\
\cdot & \cdot & \cdot & \cdot & & 1 & (-)^r (n-r)^{(r-r)} q^{-r} r_{(r)}
\end{vmatrix}
$$

which gives

$$H_r(x)(-q)^r = x^{(r)} - rp(n-r+1)x^{(r-1)} + \ldots \ldots (-)^r n^{(r)} p^r$$

$$= G_r(x)$$

and $\overset{n}{\underset{0}{S}} H_r(x) f(x)\, t^x = \overset{n}{\underset{0}{S}} t^x \Delta^r x^{(r)} f(x)$

$$= (-)^r (t-1)^r \frac{d^r}{dt^r} (pt + q)^n$$

by using (2) page 63.

$$= (-)^r (t-1)^r n^{(r)} p^r (pt+q)^{n-r}$$

Hence $\overset{n}{\underset{0}{S}} H_r(x) q^r f(x)\, x^{(s)}$

$$= (-)^r n^{(r)} p^r q^r \left[\frac{d^s}{dt^s} (t-1)^r (pt+q)^{n-r} \right]_{t=1}$$

i.e. $\overset{n}{\underset{0}{S}} G_r(x)\, G_s(x)\, f(x) = 0$ s less than r

$= n^{(r)} p^r q^r r!$ $s=r$

(Refer to 10 and 16 in 'References to Literature'.)

(c) THE RELATION

$A_oH_o(x) + A_1H_1(x) + A_2H_2(x) + A_3H_3(x) + \text{----------} A_rH_r(x) = x^{(r)}$

where $H_s(x)$ is an orthogonal polynomial of degree s with respect to the nucleus function $w(x)$.

Let the frequency generating function of $H_r(x)\ w(x)$ be $a^rF(t,r)$ .This assumption implies that $f(x)$ exists such that

$$H_r(x)\ w(x) = (-)^r \Delta^r f(x-r)$$

Since $H_r(x)$ is a polynomial of degree r in x,clearly it is always possible to reverse this relation and find an expression for $x^{(r)}$ as a linear function of the $H_s(x)$ ' s. If

$$x^{(r)} = A_oH_0(x) + A_1H_1(x) + A_2H_2(x) + \text{----} \quad + A_rH_r(x)$$

then we shall show that the A's can be expressed in terms of $F(t,r)$.

By successive multiplication of both sides of this expression by $H_o(x)\ w(x)$, $H_1(x)w(x)$, $H_2(x)w(x)$, ...etc and summing over x ,we have

$$A_0\ S\ H_0^2(x)w(x) = S\ x^{(r)}H_0(x)w(x)$$

$$A_1\ S\ H_1^2(x)w(x) = S\ x^{(r)}H_1(x)w(x)$$

$$A_2\ S\ H_2^2(x)w(x) = S\ x^{(r)}H_2(x)w(x)$$

$$A_r\ S\ H_r^2(x)w(x) = S\ x^{(r)}\ H_r(x)w(x)$$

Now $S\ H_s(x)\ t^x\ w(x) = a^s\ F(t,s)$

and so

$$S\ x^{(r)} H_s(x)\ w(x) = \frac{d^r}{dt^r}\ a^s\ F(t,s) \Big|_{a=0}$$

$$= r_{(s)}\ s!\ F^{r-s}(1,s)$$

where $F^{r-s} \equiv \dfrac{d^{r-s}}{dt^{r-s}} F(t,s)$

Further $S\ H_s^2(x)\ w(x) = S\ x^{(s)}\ H_s(x) w(x)$

$$= s!\ F(1,s)$$

where we assume that the coefficient of $x^{(s)}$ in $H_s(x)$ is unity.

Hence (1) provided the coefficient of $x^{(s)}$ in $H_s(x)$ is unity

(2) $F(1,0)$, $F(1,1)$, $F(1,2)$, ...$F(1,r)$ do not vanish

(3) $\sum_0^n H_s(x) w(x)\ x^{(r)} t^x$ when $n=\infty$, converges uniformly

for $1-\varepsilon < t < 1+\varepsilon$ $\varepsilon > 0$,

then

$$x^{(r)} = \frac{F^r(1,0)}{F(1,0)} H_0(x) + r\,\frac{F^{r-1}(1,1)}{F(1,1)} H_1(x) + r_{(2)}\frac{F^{r-2}(1,2)}{F(1,2)} H_2(x)$$

$$+ \text{------} \quad + r\,\frac{F^1(1,r-1)}{F(1,r-1)} H_{r-1}(x) + H_r(x)$$

We shall refer to this relation as expression (10)

Example 1 From the preceding section we have seen that

$$a^r e^{m(t-1)} = a^r F(t,r) \text{ generates } K_r(x)w(x) \text{ where}$$

$$K_r(x) = \frac{x^{(r)}}{m^r} - r\frac{x^{(r-1)}}{m^{r-1}} + r_{(2)}\frac{x^{(r-2)}}{m^{r-2}} -$$

$$w(x) = \frac{e^{-m}m^x}{x!}$$

and so $a^r m^r e^{m(t-1)} = a^r F_1(t,r)$ generates $m^r K_r(x)w(x)$

where $m^r K_r(x)$ has unity as coefficient of $x^{(r)}$

and $F_1(t,r) = m^r F(t,r)$

Hence :-

$$x^{(r)} = m^r K_r(x) + m^{r-1}K_{r-1}(x)rm + m^{r-2}K_{r-2}(x)m^2 r_{(2)} + \text{------------}$$

i.e.

$$\frac{x^{(r)}}{m^r} = K_r(x) + rK_{r-1}(x) + r_{(2)}K_{r-2}(x) + \text{------------} + K_0(x)$$

which is a known result for Charlier Polynomials .

Example 2

If $G_r(x)$ is the r th Gram polynomial,

then $a^r n^{(r)}p^r q^r(1+pa)^{n-r}$ generates $G_r(x)\ w(x)$ where

$w(x) = n_{(x)}\ p^x q^{n-x}$ (This is proved on p.75)

Since $G_r(x)$ has unit coefficient for $x^{(r)}$,

then $F(t,r) = n^{(r)}p^r q^r(\ pt+\ q)^{n-r}$ and substituting in (10)

$$x^{(r)} = G_r + rp(n-r+1)G_{r-1} + r_{(2)}p^2(n-r+2)^{(2)}G_{r-2} +$$

$$+ r_{(2)}(n-2)^{(r-2)}p^{r-2}G_2 + r(n-1)^{(r-1)}p^{r-1}G_1 + p^r n^{(r)}$$

which is a new result to be compared with

$$x^{(r)} = G_r + rp(n-r+1)G_{r-1}(x,n-1) + r_{(2)}p^2(n-r+1)^{(2)}G_{r-2}(x,n-2) + \ldots\ldots$$

obtained by Aitken and Gonin ,ref.10.

Example 3 It is known, and indeed could be demonstrated by the determinantal form of $H_r(x)$ on page 71, that the factorial moment generating function of the hypergeometric orthogonal polynomials $U_r(x)\Psi(x)$ is

$$a^r F(t,r)=\frac{n^{(r)}(Np)^{(r)}(N-n)^{(r)}(Nq)^{(r)}}{N^{(2r)}\,(N-r+1)^{(r)}}\; a^r F(-n+r,-Np+r,-N+2r;\,1-t)$$

where F is a degenerate hypergeometric series. Since $U_r(x)$ has unit coefficient of $x^{(r)}$, we have

$$x^{(r)} = U_r(x) + \frac{r(n-r+1)(Np-r+1)}{N-2r+2}U_{r-1}(x) + r_{(2)}\frac{(n-r+2)^{(2)}(Np-r+2)^{(2)}}{(N-2r+4)^{(2)}}U_r(x)$$

$$+r_{(3)}\frac{(n-r+3)^{(3)}(Np-r+3)^{(3)}}{(N-2r+6)^{(3)}}U_{r-3}(x) + \ldots\ldots\ldots\ldots\ldots\ldots$$

where

$$U_r(x) = x^{(r)} - \frac{r(n-r+1)(Np-r+1)}{(N-2r+2)}x^{(r-1)} + r_{(2)}\frac{(n-r+2)^{(2)}(Np-r+2)^{(2)}}{(N-2r+3)^{(2)}}x^{(r-2)}$$

It is to be observed that the usual method of obtaining this type of relation by the application of symbolic algebra, for example, from $G_r(x) = (1+p\Delta)^{-n+r-1}x^{(r)}$ we have $(1+p\Delta)^{n-r+1}G_r(x)= x^{(r)}$ which on expansion of the L.H.S. and using $\Delta G_r(x)= rG_{r-1}(x,n-1)$ leads to the result given on the previous page, is not applicable to the hypergeometric case since it is not obvious what the reciprocal of $U_r(x) = F(n-r+1,Np-r+1,N-2r+2;\,-\Delta)\,x^{(r)}$ is.

(c) Continued

GENERAL ORTHOGONAL POLYNOMIAL RECURRENCE FORMULA.

As before, we assume that $a^r F(t,r)$ generates $H_r(x)\ w(x)$ where $H_r(x)$ are orthogonal polynomials with respect to $w(x)$. Now $H_1(x)\ H_r(x)$ is of degree $r+1$ and thus we are justified in assuming

$$H_r(x)H_1(x) = A_0H_0(x) + A_1H_1(x) + A_2H_2(x) + \ldots\ldots + A_{r+1}H_{r+1}(x) \qquad \ldots..(11)$$

Clearly $S\ H_r(x)\ H_s(x)w(x) = S\ x^{(s)}\ H_r(x)\ w(x) = 0$ s less than r

$$= \left[\frac{d^s}{dt^s}\ a^s\ F(t,r)\right]_{a=0}$$

where again we assume $H_r(x) = x^{(r)} + A\ x^{(r-1)} + B\ x^{(r-2)} + \ldots$

Evidently

$$S\ H_r(x)\ H_1(x)\ H_s(x)\ w(x) = 0 \quad \text{s less than r-1}$$

and so $A_0=A_1=A_2=A_3= \ldots\ \ldots = A_{r-2} = 0$

and

$$A_{r-1}\ S\ H^2_{r-1}(x)\ w(x) = S\ H_r(x)\ H_{r-1}(x)\ H_1(x)\ w(x)$$

$$A_r\ S\ H^2_r(x)\ w(x) = S\ H^2_r(x)\ H_1(x)\ w(x)$$

$$A_{r+1}\ S\ H^2_{r+1}(x)\ w(x) = S\ H_r(x)\ H_{r+1}(x)\ H_1(x)\ w(x)$$

by multiplying both sides of (11) by $H_{r-1}(x)w(x)$, $H_r(x)\ w(x)$, $H_{r+1}(x)\ w(x)$ and summing over x.

It is easy to see that $A_{r+1} = 1$ and $A_{r-1} = r\ F(1,r)/F(1,r-1)$

For A_r we refer to the relation established in (10), namely

$$x^{(r)} = H_r(x) + r\frac{F'(1,r-1)}{F(1,r-1)}\ H_{r-1}(x) + \ldots\ \ldots + H_0(x)$$

and thence

$$S\ x^{(r)}H_r(x)\ H_1(x)\ w(x) = SH^2_r(x)H_1(x)w(x) + r\frac{F'(1,r-1)}{F(1,r-1)}SH_rH_{r-1}H_1\ w(x)$$

i.e.

$$S\, H_r^2(x)H_1(x)w(x) = SH_r(x)w(x)\quad x^{(r)}\left(x - \frac{F'(1,0)}{F(1,0)}\right) - rF'\frac{(1,r-1)}{F(1,r-1)}F(1,r)r!$$

$$= r - \frac{F'(1,0)}{F(1,0)}\quad r!\; F(1,r) + (r+1)r!\; F'(1,r) - r\frac{F'(1,r-1)}{F(1,r-1)}\, F(1,r)$$

Hence we have the recurrence relation :-

$$H_{r+1} - \left[x - r - (r+1)\frac{F'(1,r)}{F(1,r)} + r\frac{F'(1,r-1)}{F(1,r-1)}\right] H_r \;+\; r\frac{F(1,r)}{F(1,r-1)}\; H_{r-1} = 0$$

This recurrence formula is one of a series of increasing difficulty which can be found, the others being $H_r(x)\, H_2(x)$, $H_r(x)\, H_3(x)$, ...$H_r(x)\, H_s(x)$. They are useful in the development of polynomials in two variables which we shall discuss in Chapter 111. The expression for $H_r(x)\, H_2(x)$ is given here:-

$$H_r(x)\, H_2(x) = H_{r+2}(x) + \left[2r - 2\frac{F'(1,1)}{F(1,1)} + (r+2)\frac{F'(1,r+1)}{F(1,r+1)} - r\frac{F'(1,r-1)}{F(1,r-1)}\right] H_{r+1}(x) \quad + \left[(r+1)\frac{F(1,r+1)}{F(1,r)} + r\frac{F(1,r)}{F(1,r-1)} \cdot \frac{F(1,1)}{F(1,0)} + \right.$$

$$\left(r - \frac{F'(1,0)}{F(1,0)} + (r+1)\frac{F'(1,r)}{F(1,r)} - r\frac{F'(1,r-1)}{F(1,r-1)}\right)\left(r - 1 + \frac{F'(1,0)}{F(1,\theta)} - \right.$$

$$\left.\left. 2\frac{F'(1,1)}{F(1,1)} + (r+1)\frac{F'(1,r)}{F(1,r)} - r\frac{F'(1,r-1)}{F(1,r-1)}\right)\right] H_r(x)$$

$$+ \; r\frac{F(1,r)}{F(1,r-1)}\left[2r - 2 + (r+1)\frac{F'(1,r)}{F(1,r)} - (r-1)\frac{F'(1,r-2)}{F(1,\, r-2)} - 2\frac{F'(1,1)}{F(1,1)}\right] H_{r-1}(x)$$

$$+ \; r^{(2)}\frac{F(1,r)}{F(1,r-2)}\; H_{r-2}(x) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

We turn now to some applications of these recurrence results.

Application 1 Charlier Polynomials.

In this case $F(t,r) = m^r e^{m(t-1)}$ and

$m^r a^r e^{m(t-1)}$ generates $\left(x^{(r)} - rmx^{(r-1)} + r_{(2)} m^2 x^{(r-2)} - \ldots \right) w(x)$

$$= m^r K_r(x)\, w(x)$$

$$= K'_r(x)\, w(x)$$

Insert $F(t,r) = m^r e^{m(t-1)}$ in (12) and we have

$$K'_{r+1}(x) + (r+m-x)\, K'_r(x) + rmK'_{r-1}(x) = 0$$

i.e.

$$K'_{r+1}(x)/m^{r+1} + \frac{(r-x+m)}{m} K'_r(x)/m^r + r\, K'_{r-1}(x)/m^r = 0$$

and so

$$K_{r+1}(x) + \left(\frac{r-x}{m} + 1\right) K_r(x) + \frac{r}{m} K_{r-1}(x) = 0$$

This can be compared with

$$K_{r+1}(x) - K_1(x) K_r(x-1) + \frac{r}{m} K_{r-1}(x-1) = 0$$

obtained by Aitken, Proc. Roy. Soc. Ed. Vol. L11 .

Application 2 Gram Polynomials.

Take $F(t,r) = n^{(r)} p^r q^r (pt+q)^{n-r}$ and then

$$\frac{F'(1,r)}{F(1,r)} = (n-r)p \qquad \frac{F'(1,r-1)}{F(1,r-1)} = (n-r+1)p$$

$$\frac{F(1,r)}{F(1,r-1)} = (n-r+1)pq$$ and inserting these in (12)

we have

$$G_{r+1}(x) + \Big[r-x+(r+1)(n-r)p - r(n-r+1)p \Big] G_r(x) + r(n-r+1)pqG_{r-1}(x) =$$

i.e.

$$G_{r+1}(x) + \Big[r-x+(n-2r)p \Big] G_r(x) + r(n-r+1)pqG_{r-1}(x) = 0$$

which is a known result although otherwise difficult to obtain.

(N.B.

We may remark that the recurrence formula for the Charlier polynomials obtained above follows from that for Gram's polynomials by making $n\to\;$, and $p\to 0$ in such a way that $np\to m$.)

Application of the Second Recurrence Formula. Gram Polynomials.

With $F(t,r)$ defined as in application 2 , we easily obtain the following :-

$F'(1,r-2)/F(1,r-2) = (n-r+2)p$. $F'(1,1)/F(1,1) = (n-1)p$

$F(1,r+1)/F(1,r) = (n-r)pq$ $F(1,r)/F(1,r-1) = (n-r+1)pq$

$F(1,1)/F(1,0) = npq$. $F'(1,0)/F(1,0) = np$ $F'(1,1)/F(1,1) = (n-1)p$

$F(1,r)/F(1,r-2) = (n-r+2)^{(2)}p^2q^2$ $F'(1,r+1)/F(1,r+1) = (n-r-1)p$

Substitute these values in (13) and we obtain :-

$$G_r(x)G_2(x) = G_{r+2}(x) + \Big[2r-2(n-1)p+(r+2)(n-r-1)p-r(n-r+1)p\Big]G_{r+1}(x)$$
$$+\Big[(r+1)(n-r)pq+r(n-r+1)pq-npq+\Big(r-np+(r+1)(n-r)p-r(n-r+1)p\Big)\Big(r-1+np-2(n-1)p+(r+1)(n-r)p-r(n-r+1)p\Big)\Big]G_r(x)$$
$$+r(n-r+1)pq\Big[2r-2+(r+1)(n-r)p-(r-1)(n-r+2)p-2(n-1)p\Big]G_{r-1}(x)$$
$$+r^{(2)}(n-r+2)^{(2)}p^2q^2G_{r-2}(x)$$

and this becomes, with some simplification ,

$$G_r(x)\,G_2(x) = G_{r+2}(x) + 2r(1-2p)G_{r+1}(x) + \Big[2rpq(n-r)+r^{(2)}(1-2p)^2\Big]G_r(x)$$
$$+2r^{(2)}pq(n-r+1)(1-2p)G_{r-1}(x) + r^{(2)}p^2q^2\,(n-r+2)^{(2)}G_{r-2}(x)$$

Application 3. Hypergeometric Polynomials.

Here $$F(t,r) = \frac{n^{(r)}(Np)^{(r)}(N-n)^{(r)}(Nq)^{(r)}}{N^{(2r)}\,(N-r+1)^{(r)}}F(-n+r,-Np+r,-N+2r;\,-a)$$

where $F(-n+r,-Np+r,-N+2r:-a)$ is a degenerate Hypergeometric series.

We easily find $F'(1,r)/F(1,r) = (n-r)(Np-r)/(N-2r)$

$F'(1,r-1)/F(1,r-1) = (n-r+1)(Np-r+1)/(N-2r+2)$

$$\frac{F(1,r)}{F(1,r-1)} = \frac{(n-r+1)(Np-r+1)(N-n-r+1)(Nq-r+1)(N-r+2)}{(2)(N-2r+2)\quad(2)(N-2r+3)}$$

and so the recurrence formula for the 'hypergeometric' polynomials is :-

$$U_{r+1}(x) + \left[r-x+ \frac{(r+1)(n-r)(Np-r)}{N-2r} - r\frac{(n-r+1)(Np-r+1)}{N-2r+2} \right] U_r(x)$$

$$+ r\frac{(n-r+1)(Np-r+1)(N-n-r+1)(Nq-r+1)(N-r+2)}{(N-2r+3)(N-2r+2)(N-2r+2)(N-2r+1)} U_{r-1}(x) = 0$$

RECURRENCE RELATION IN THE CONTINUOUS CASE.

In the continuous case suppose

$$a^r W(a,r) = \int_c^b H_r(x)\, w(x)\, e^{ax}\, dx \quad \text{where } H_r(x)$$

is a polynomial in x of degree r, orthogonal with respect to $w(x)$ over the range $(c\ b)$.

Then if

$$H_r(x)\, H_1(x) = A_0H_0(x) + A_1H_1(x) + A_2H_2(x) + \dots + A_{r+1}H_{r+1}(x)$$

$$\int_c^b H_r(x)H_1(x)H_s(x)w(x)dx \equiv 0 \quad \text{for s less than r-1.}$$

and so $A_0 = A_1 = A_2 = A_3 = \dots \dots \dots = A_{r-2} = 0$

Again

$$\int_c^b H_r(x)H_1(x)H_{r-1}(x)w(x)\, dx = A_{r-1}\int_c^b H^2_{r-1}(x)w(x)dx$$

$$\int_c^b H_r(x)H_1(x)H_r(x)\, w(x)dx = A_r \int_c^b H^2_r(x)w(x)dx$$

$$\int_c^b H_r(x)H_1(x)H_{r+1}(x)w(x)dx = A_{r+1}\int_c^b H_{r+1}^2(x)w(x)\,dx$$

<u>Case 1</u> Clearly since $a^r W(a,r) = \int_c^b H_r(x)w(x)e^{ax}\,dx$

and (a) c and b are both finite , (b) $H_r(x)w(x)e^{ax}\,x^s$ is a continuous function of x and a provided $w(x)$ is a continuous function of x , (c) $H_r(x)w(x)e^{ax}$ is integrable , then

$$\frac{d^s}{da^s}\,a^r W(a,r) = \frac{d^s}{da^s}\int_c^b H_r(x)w(x)e^{ax}\,dx$$

$$= \int_c^b \frac{d^s}{da^s}\,H_r(x)w(x)e^{ax}\,dx$$

$$= \int_c^b H_r(x)w(x)x^s\,e^{ax}\,dx$$

Without loss of generality we can take r greater than or equal to s, and then

$$\int_c^b H_r(x)\,H_s(x)w(x)e^{ax}\,dx = \int_c^b H_r(x)w(x)e^{ax}\,(\,x^s + Ax^{s-1} + Bx^{s-2} + .\,)dx$$

Hence $\int_c^b H_r(x)H_1(x)H_{r+1}(x)w(x)dx = \int_c^b H_{r+1}(x)x^{r+1}\,w(x)\,dx$

$$= \left[\frac{d^{r+1}}{da^{r+1}}\,a^{r+1}\,W(a,r+1\,)\right]_{a=0}$$

$$= (r+1)!\;W(0,r+1)$$

and so $A_{r+1} = 1$

Further $\int_c^b H_r(x)H_1(x)H_{r-1}(x)w(x)\,dx = \int_c^b H_r(x)\,x^r w(x)\,dx$

$$= \frac{d^r}{da^r}\int_c^b H_r(x)w(x)e^{ax}\,dx \Bigg|_{a=0}$$

$$= r!\,W(0,r)$$

and so $\quad A_{r-1} = \dfrac{r!\,W(0,r)}{(r-1)!\,W(0,r-1)} = r\,\dfrac{W(0,r)}{W(0,r-1)}$

Using
$$x^r = H_r(x) + r\frac{W'(0,r-1)}{W(0,r-1)}H_{r-1}(x) + r_{(2)}\frac{W''(0,r-2)}{W(0,r-2)}H_{r-2}(x) + \cdot$$

we have
$$A_r\,r!\,W(0,r) = \int_c^b H_r(x)\,H_1(x)\left\{x^r - r\frac{W'(0,r-1)}{W(0,r-1)}H_{r-1}(x) + \ldots\right\}w(x)\,dx$$

$$= \int_c^b H_r(x)\,x^r\left(x - \frac{W'(0,0)}{W(0,0)}\right) - r\frac{W'(0,r-1)}{W(0,r-1)}H_r(x)H_{r-1}(x)\left(x - \frac{W'(0,0)}{W(0,0)}\right)w(x)\,dx$$

$$= (r+1)r!\,W'(0,r) - \frac{W'(0,0)}{W(0,0)}r!\,W(0,r) - r\frac{W'(0,r-1)}{W(0,r-1)}r!\,W(0,r)$$

Hence $\quad H_r(x)\,H_1(x) = H_{r+1}(x) + r\dfrac{W(0,r)}{W(0,r-1)}H_{r-1}(x) +$

$$\left[(r+1)\frac{W'(0,r)}{W(0,r)} - \frac{W'(0,0)}{W(0,0)} - r\frac{W'(0,r-1)}{W(0,r-1)}\right]H_r(x)$$

i.e.

$$H_{r+1}(x) + \left[(r+1)\frac{W'(0,r)}{W(0,r)} - r\frac{W'(0,r-1)}{W(0,r-1)} - x\right]H_r(x) + r\frac{W(0,r)}{W(0,r-1)}H_{r-1}(x) = 0 \qquad \ldots\ldots(14)$$

Case (ii) when the limits of integration are infinite. To begin with assume c remains finite. Then

$$\frac{d^s}{da^s}\, a^r\, W(a,r) = \int_c^{\infty} H_r(x)\, w(x)\, \frac{d^s}{da^s}\, e^{ax}\, dx$$

if (a) $H_r(x)\, w(x)\, x^s e^{ax}$ is a continuous function of x and 'a' which is true if w(x) is a continuous function of x (b) $\int_c^{\infty} H_r(x) w(x)\, x^s\, e^{ax}\, dx$ converges uniformly with respect to 'a'

(c) $\int_c^{\infty} H_r(x) w(x) e^{ax}\, dx$ converges .

If these conditions are satisfied ,the recurrence formula proof follows as in case (i).

(N.B. The result quoted above ,namely

$$x^r = H_r(x) + r\frac{W'(0,r-1)}{W(0,r-1)}H_{r-1}(x) + r_{(2)}\,\frac{W''(0,r-2)}{W(0,r-2)}H_{r-2}(x) + \ldots\ldots\ldots$$

corresponds to formula (10) in the discrete case i.e. to

$$x^{(r)} = \frac{F^r(1,0)}{F(1,0)}\, H_0(x) + r\frac{F^{r-1}(1,1)}{F(1,1)}H_1(x) + \ldots\; + r\frac{F'(1,r-1)}{F(1,r-1)}H_{r-1}(x) + H_r(x)$$

and the proof proceeds in a similar way.)

Application.

Consider $W(a,r) = e^{\frac{1}{2}a^2}$ when $H_r(x)$ becomes the r th.

Hermite polynomial. In this case $b=\infty, c=-\infty$, and $w(x) = e^{-\frac{1}{2}x^2}$ and (a) $H_r(x)w(x)x^s e^{ax}$ is a continuous function of x and 'a', (b) $\int_{-\infty}^{\infty} H_r(x)w(x)x^s e^{ax}\,dx = \int_{-\infty}^{\infty} H_r(x)x^s e^{ax-\frac{1}{2}x^2}\,dx$ must be uniformly convergent with respect to a . Now

$$\int_0^{\infty} H_r(x)\, x^s\, e^{ax-\frac{1}{2}x^2}\,dx = e^{\frac{1}{2}a^2}\int_0^{\infty} H_r(x)x^s\, e^{-\frac{1}{2}(a-x)^2}\,dx$$

If $-1 < a < +1$

$$\left|\int_{\lambda}^{\infty} H_r(x)\, x^s\, e^{ax-\frac{1}{2}x^2}\,dx\right| < e^{\frac{1}{2}}\left|\int_{\lambda}^{\infty} H_r(x)\, x^s\, e^{-\frac{1}{2}(a-x)^2}\,dx\right|$$

$$< A\, e^{\frac{1}{2}}\left|\int_{\lambda}^{\infty} x^{r+s}\, e^{-\frac{1}{2}(x-1)^2}\,dx\right|$$

where A depends on r only ,since

$$H_r(x) = x^r - \frac{r(r-1)}{2\ 1!}\, x^{r-2} + \frac{r(r-1)(r-2)(r-3)}{2^2\ 2!}\, x^{r-4} - \ldots\ldots\ldots$$

and

$$\left|H_r(x)\right| < x^r\left(1 + \frac{r^{(2)}}{2.\ 1!} + \frac{r^{(4)}}{2^2.2!} + \ldots\ldots\ldots\ldots\ldots\right) \qquad x \geqslant 1$$

Hence $\left|\int_{\lambda}^{\infty} H_r(x)\, x^s\, e^{ax-\frac{1}{2}x^2}\,dx\right| < e'$

for all values of $\lambda \geqslant X$ independent of a ,for $\int_{\lambda}^{\infty} x^{r+s}\, e^{-\frac{1}{2}(x-1)^2}\,dx$ can be made as small as we please by taking λ large enough ,since $e^{-\frac{1}{2}(x-1)^2} \to 0$ as $x \to \infty$ more rapidly than any power of x.

(c) $\int_{-\infty}^{\infty} H_r(x)\, e^{-\frac{1}{2}x^2+ax}\,dx$ converges as shown in (b) .

Hence we can apply formula (14) ,and noting that

$$\frac{W'(0,r)}{W(0,r)} = \frac{W'(0,r-1)}{W(0,r-1)} = 0 \qquad \frac{W(0,r)}{W(0,r-1)} = 1$$

we have

$$\underline{H_{r+1}(x) - x\,H_r(x) + r\,H_{r-1}(x) = 0}$$

which is the usual recurrence result for the Hermite polynomials.

(d) THE GENERATING FUNCTION OF $w(x)\, f(\Delta)\, x^{(r)}$

It is a curious fact that orthogonal polynomials can be expressed elegantly and concisely by means of symbolic operators. We recall the following examples, and in each case note the factorial moment generating function of the product of the polynomial and the nucleus function. We consider those of Charlier, Gram and the 'Hypergeometric' .

Polynomial	$= H_r(x)$	F.M.G.F. of $H_r(x)\, w(x)$
Charlier	$K_r(x) = e^{-m\Delta} x^{(r)}$	$a^r e^{ma}$
Gram	$G_r(x) = (1+p\Delta)^{-n+r-1} x^{(r)}$	$n^{(r)} p^r q^r\, a^r (1+pa)^{n-r}$

Hypergeometric

$U_r(x) = F(n-r+1, Np-r+1, N-2r+2 : -\Delta)\, x^{(r)}$

$A\, a^r F(-n+r, -Np+r, -N+2r : a)$

where A is a function of n, r, N and p only

This suggests the form $f(\Delta)\,x^{(r)}$ for a general polynomial of degree r.We give an attempt to find the frequency generating function of $w(x)\,f(\Delta)x^{(r)}$ .To obtain this we make use of Laplace's Integral for $x_{(s)}$ namely

$$\frac{x(x-1)(x-2)(x-3)\,\ldots\ldots\ldots(x-s+1)}{s!} = \frac{1}{2\,\Pi\, i}\int_0^{(+1)} \frac{t^x}{(t-1)^{s+1}}\,dt$$

the path of integration being a loop which starts from the origin ,makes a positive circuit around t=1, and returns to the origin. Now if W(t) is the frequency generating function of w(x) then

$$W(t) = \overset{n}{\underset{0}{S}}\; t^x\, w(x) \text{ where n is finite or infinite.}$$

and
$$\frac{s!}{2\Pi\, i}\int_0^{(+1)} \frac{W(zt)\,dz}{(z-1)^{s+1}} = \frac{s!}{2\,\Pi\, i}\int_0^{(+1)} \frac{\overset{x}{S}\,(tz)\;\;dz}{(z-1)^{s+1}}\,w(x) .$$

$$= \frac{s!}{2\,\Pi\, i}\; S \int_0^{(+1)} t^x z^x\, w(x)\, dz$$

$$= S\;\; t^x\, x^{(s)}\;\; w(x)$$

provided $S\;\dfrac{(tz)^x w(x)}{(z-1)^{s+1}}$ is a uniformly convergent series of continuous functions with respect to z.

If this condition is satisfied,

$$S\ t^x w(x) f(\Delta) x^{(r)} = S\ t^x w(x)\left[a_0 + a_1\Delta + a_2\Delta^2 + a_3\Delta^3 + \ldots\ldots + a_r\Delta^r + a_{r+1}\Delta^{r+1} + a_{r+2}\Delta^{r+2} \ldots\ldots\right]$$

$$= \frac{r!}{2\Pi i}\int_0^{(+1+)} W(zt)\left[\frac{a_0}{(z-1)^{r+1}} + \frac{a_1}{(z-1)^r} + \quad \frac{a_r}{(z-1)}\right] dz$$

$$= \frac{r!}{2\Pi i}\int_0^{(+1)} \frac{W(zt)}{(z-1)^{r+1}}\left[f(z-1) - a_{r+1}(z-1)^{r+1} - a_{r+2}(z-1)^{r+2} - \quad\right] dz$$

$$= \frac{r!}{2\Pi i}\int_0^{(+1)} \frac{W(zt) f(z-1) dz}{(z-1)^{r+1}} - \frac{r!}{2\Pi i}\int_0^{(+1)} \frac{W(zt)}{(z-1)^{r+1}} \overset{\infty}{\underset{r+1}{S}} a_k (z-1)^k dz$$

$$= \frac{r!}{2\Pi i}\left[\int_0^{(+1)} \frac{W(zt) f(z-1) dz}{(z-1)^{r+1}} - \overset{\infty}{\underset{r+1}{S}} \int_0^{(+1)} \frac{a_k (z-1)^k W(zt)}{(z-1)^{r+1}} dz\right]$$

$$= \frac{r!}{2\Pi i}\int_0^{(+1)} \frac{W(zt) f(z-1) dz}{(z-1)^{r+1}}$$

provided W(zt) is analytic within and on the path of integration and $\overset{\infty}{\underset{r+1}{S}}\ a_k \frac{(z-1)^k\ W(zt)}{(z-1)^{r+1}}$ is uniformly convergent.

When these conditions are satisfied

$$S\ t^x w(x) f(\Delta) x^{(r)} = \frac{r!}{2\Pi i}\int_0^{(+1)} \frac{W(zt) f(z-1)}{(z-1)^{r+1}} dz \quad \ldots\ldots\ldots\ldots (15)$$

Again if $S\, x^{(s)} w(x)\, t^x f(\Delta)\, x^{(r)} = 0$ s less than r

$\neq 0$ s=r

then $f(\Delta)x^{(r)}$ are orthogonal polynomials with regard to $w(x)$.

$$\text{i.e. } \frac{d^s}{dt^s} \int_0^{(+1)} \frac{W(zt)f(z-1)}{(z-1)^{r+1}}\, dz = 0 \text{ for all s less than r and } t=1.$$

Application 1

$$\text{Take } W(t) = e^{m(t-1)} \text{ so that } w(x) = \frac{e^{-m} m^x}{x!}$$

Then $W(zt)$ is analytic within and on the path of integration described above and if

$$f(z-1) = a_0 + a_1(z-1) + a_2(z-1)^2 + a_3(z-1)^3 + \ldots\ldots\ldots$$

then

$$\int_0^{(+1)} \frac{W(zt)f(z-1)dz}{(z-1)^{r+1}} = \int_0^{(+1)} \frac{e^{m(zt-1)}\, dz}{(z-1)^{r+1}} \left[a_0 + a_1(z-1) + a_2(z-1)^2 + \ldots\ldots\ldots\right]$$

$$= e^{m(t-1)} \int_0^{(+1)} \frac{e^{m(z-1)t}\, dz}{(z-1)^{r+1}} \left[a_0 + a_1(z-1) + a_2(z-1)^2 + \right]$$

$$= e^{m(t-1)}\, 2\,\Pi\, i\, \frac{(mt)^r}{r!} \left\{a_0 + \frac{(mt)^{r-1}}{(r-1)!} a_1 + \frac{(mt)^{r-2}}{(r-2)!} a_2 + \ldots\ldots \right\}$$

$$= 2\Pi\, i\, e^{m(t-1)} \frac{m^r}{r!} \left[t^r a_0 + \frac{r t^{r-1}}{m} a_1 + \frac{r^{(2)} t^{r-2}}{m^2} a_2 + \ldots \right]$$

since if $f(z)$ has a pole of order m at z=a, and is analytic throughout the contour C except at z=a , then

$$\int_C f(z)\, dz = 2\Pi\, i\, R \text{ where } R \text{ is the residue at } z=a.$$

Hence the condition $\frac{d^s}{dt^s}\int_0^{(+1)} \frac{W(zt)f(z-1)dz}{(z-1)^{r+1}} = 0$ (s less than r)

leads to

$$a_0 + \frac{r}{m}a_1 + \frac{r^{(2)}}{m^2}a_2 + \cdots\cdots = 0$$

$$a_0 + \frac{(r-1)}{m}a_1 + \frac{(r-1)^{(2)}}{m^2}a_2 + \cdots\cdots = 0$$

$$a_0 + \frac{(r-2)}{m}a_1 + \frac{(r-2)^{(2)}}{m^2}a_2 + \cdots\cdots = 0$$

$$\vdots$$

$$a_0 + \frac{2a_1}{m} + \frac{2!}{m^2}a_2 = 0$$

$$a_0 + \frac{a_1}{m} = 0$$

and so

$$a_1 = -ma_0 \quad a_2 = \frac{m^2}{2!}a_0 \quad a_3 = \frac{m^3}{3!}a_0$$

and so

$$f(z) = a_0\left(1 - mz + \frac{m^2z^2}{2!} - \frac{m^3z^3}{3!} + \cdots\cdots\right)$$

$$= a_0 e^{-mz}$$

Hence $e^{-m\Delta} x^{(r)}$ are orthogonal polynomials with respect to $w(x) = \frac{e^{-m} m^x}{x!}$

Application 2 Take $W(zt) = (ptz+q)^n$ and $w(x) = n_{(x)} p^x q^{n-x}$

Then $f(\Delta)\, x^{(r)}$ are orthogonal polynomials with respect to $w(x)$ provided $\frac{d^s}{dt^s}\int_0^{(+1)} \frac{(ptz+q)^n f(z-1)}{(z-1)^{r+1}}dz = 0$ s less than r. $t=1$

Assume, as before, $f(z-1) = a_0 + a_1(z-1) + a_2(z-1)^2 + \dots$.

then $$\frac{d^s}{dt^s} \int_0^{(+1)} \frac{\left[pt(z-1)+Pt+q\right]^n}{(z-1)^{r+1}} \left[a_0 + a_1(z-1) + a_2(z-1)^2 + \dots \right] dz$$

$$= 0 \text{ for } s \text{ less than } r \quad t = 1$$

The equations for $a_0, a_1, a_2, \dots\dots a_r$ are :-

$$n^{(r)} p^r a_0 + rn^{(r-1)} p^{r-1} a_1 + r^{(2)} n^{(r-2)} p^{r-2} a_2 + \dots\dots = 0$$

$$(n-1)^{(r)} p^r a_0 + r(n-1)^{(r-1)} p^{r-1} a_1 + r^{(2)} (n-1)^{(r-2)} p^{r-2} a_2 + \quad = 0$$

$$(n-2)^{(r)} p^r a_0 + r(n-2)^{(r-1)} p^{r-1} a_1 + r^{(2)} (n-2)^{(r-2)} p^{r-2} a_2 + \quad = 0$$

$$\dots \qquad \dots \qquad \dots \qquad .$$

$$(n-r+1)^{(r)} p^r a_0 + r(n-r+1)^{(r-1)} p^{r-1} a_1 + r^{(2)} (n-r+1)^{(r-2)} p^{r-2} a_2 = 0$$

and from these we find

$$a_1 = -(n-r+1)pa_0 \qquad a_2 = (n-r+2)^{(2)} a_0 p^2/2! \qquad a_3 = -(n-r+3)^{(3)} \frac{p^3}{3!} a_0$$

and so

$$f(z) = 1 - p(n-r+1)z + \frac{p^2}{2!}(n-r+2)^{(2)} z^2 - \dots\dots$$

$$= (1+pz)^{-(n-r+1)}$$

Hence $(1 + p\Delta)^{-(n-r+1)} x^{(r)}$ are orthogonal polynomials with respect to $w(x) = n_{(x)} p^x q^{n-x}$

(e)

A NEW APPROACH TO GRAM POLYNOMIALS.

It has been observed that $e^{-m} m^x K_r(x) = \frac{d^r}{dm^r}(e^{-m} m^x)$

where $K_r(x)$ is the r th Charlier polynomial. From this fact the orthogonal property and recurrence relation can be obtained without difficulty. In this section we show a similar property with regard to Gram's polynomials and demonstrate the orthogonal property and other known results.

We have $\quad (pt+q)^n = \overset{n}{\underset{0}{S}} \; w(x)\, t^x \qquad$ where $w(x)=n_{(x)}p^x q^{n-x}$

Differentiate both sides r times with respect to p, and replace t by 1 + a, then

$$a^r n^{(r)}(1+pa)^{n-r} = \overset{n}{\underset{0}{S}} \; t^x \frac{d^r}{dp^r} n_{(x)} p^x q^{n-x}$$

$$= \overset{n}{\underset{0}{S}} \; t^x n_{(x)} (-)^r \Big[(n-x)^{(r)} p^x q^{n-x-r} - r(n-x)^{(r-1)} x p^{x-1} q^{n-x-r+1} + \dots\dots\dots \Big]$$

$$= \overset{n}{\underset{0}{S}} \; t^x n_{(x)} p^{x-r} q^{n-x-r} (-)^r \Big[(n-x)^{(r)} p^r - r(n-x)^{(r-1)} x p^{r-1} q + \dots\dots\dots \Big]$$

i.e.

$$n^{(r)} a^r (1+pa)^{n-r} = S\, t^x (pq)^{-r} w(x) (-)^r \Big[(n-x)^{(r)} p^r - r(n-x)^{(r-1)} p^{r-1} q \dots \Big]$$

$$= S\, t^x w(x)\, (pq)^r\, M_r(x)$$

where $M_r(x)$ is a polynomial of degree r in x.

But $\quad S t^{x-s} w(x)\, (pq)^{-r} M_r(x)\, x^{(s)} = \frac{d^s}{da^s} n^{(r)} a^r (1+pa)^{n-r}$

and putting t=1 we have

$$S\, M_r(x)\, x^{(s)} w(x) = (pq)^r \Big[\frac{d^s}{da^s} \; n^{(r)} \; a^r (1+pa)^{n-r} \Big]_{a=0}$$

$$= 0 \qquad \text{s less than r}$$

$$= (pq)^r\, n^{(r)} r! \qquad s = r$$

By using the frequency generating function of $w(x) = n_{(x)} p^x q^{n-x}$ we have found that the polynomials $M_r(x)$ satisfy

$$\mathop{S}_{0}^{n} x^{(s)} M_r(x)\, w(x) = 0 \qquad \text{s less than r}$$

$$= n^{(r)} p^r q^r\, r!$$

i.e. $$\mathop{S}_{0}^{n} M_r(x)\, M_s(x)\, w(x) = 0 \qquad \text{s less than r}$$

$$= n^{(r)} p^r q^r r! \qquad s=r$$

i.e. $M_r(x)$ are orthogonal polynomials with respect to $w(x)$

and $$w(x)\, M_r(x) = p^r q^r \frac{d^r}{dp^r}\, w(x,p) \quad \dots\dots\dots\dots\dots\dots(16)$$

It is clear from the conditions satisfied by $M_r(x)$ that it is identical with the rth Gram polynomial. We shall now give proofs of the usual properties of these polynomials.

(1) <u>The Factorial Moment Generating Function of $G_r(x)\, w(x)$.</u>

Since $\mathop{S}_{0}^{n} t^x\, w(x)\, G_r(x)\, (pq)^{-r} = n^{(r)}\, a^r (1+pa)^{n-r}$

we have the result:-

The F.M.G.F. of $G_r(x)\, w(x)$ is $n^{(r)} (pq)^r\, (1+pa)^{n-r}$

(2) <u>Recurrence formula.</u>

From (16) we find that

$$M_r(x) = G_r(x) = x^{(r)} - rp(n-r+1)x^{(r-1)} + \dots \dots\dots\dots$$

and so $$\frac{d}{dp} G_r(x) = -r(n-r+1) G_{r-1}(x)$$

Further $$p^{x-r} q^{n-x-r}\, G_r(x) = \frac{d^r}{dp^r}\, p^x q^{n-x}$$

Differentiate both sides with respect to p and we have

$$\left[(x-r)p^{x-r-1}q^{n-x-r} - (n-x-r)p^{x-r}q^{n-x-r-1}\right]G_r(x)$$

$$- r(n-r+1)\,p^{x-r}q^{n-x-r}\,G_{r-1}(x) = \frac{d^{r+1}}{dp^{r+1}}\,p^x q^{n-x}$$

$$= p^{x-r-1}q^{n-x-r-1}G_{r+1}(x)$$

i.e.

$$\left[(x-r)p^{-1} - (n-x-r)q^{-1}\right]G_r(x) - r(n-r+1)G_{r-1}(x) = (pq)^{-1}\,G_{r+1}(x)$$

i.e.

$$G_{r+1} - \left(x - np - r(1-2p)\right)G_r + rpq(n-r+1)\,G_{r-1} = 0$$

which is the usual recurrence formula for Gram's polynomials obtained in a very much quicker manner . (See Ref. 16)

(3) $G_r(x, n-s)$ in terms of $G_r(x,n)$

Write $p^x q^{n-x-s} = (p^x q^{n-x})\,q^{-s}$ and differentiate both sides r times with respect to p , substituting $p^x q^{n-x} = (pq)^r \frac{d^r}{dp^r}\,p^x q^{n-x}$. Then

$$G_r(x,n-s) = G_r(x,n) + rspG_{r-1}(x,n) + r_{(2)}(s+1)^{(2)}p^2G_{r-2}(x,n) + \ldots\ldots \tag{17/1}$$

Similarly

$$G_r(x,n) = G_r(x,n-s) - rspG_{r-1}(x,n-s) + r_{(2)}s^{(2)}p^2G_{r-2}(x,n-s)$$

$$- r_{(3)}s^{(3)}p^3\,G_{r-3}(x,n-s) + \ldots\ldots\ldots\ldots (17/2)$$

properties which are useful in the development of polynomials in two variables.

CHAPTER THREE

ORTHOGONAL POLYNOMIALS IN TWO VARIABLES

(a) The Continuous Case. Polynomials corresponding to Hermite's in one variable.

(1) Evaluation of $\int_{-\infty}^{\infty} H_r(x)H_s(x)H_p(x)\, e^{-\frac{1}{2}x^2}\, dx$

(2) The Value of (A) $\iint_{-\infty}^{\infty} H^2_{i.k}(x,y)\, e^{-(x^2-2rxy+y^2)/2(1-r^2)}\, dx\, dy$

(B) $\iint_{-\infty}^{\infty} H_{i.k}(x,y)H_{l.m}(x,y)\, e^{-(x^2-2rxy+y^2)/2(1-r^2)}\, dx\, dy$

(3) A Symbolic Form for $H_{i.k}\,(x,y)$

(4) An Approach by Generating Functions and their Derivatives.

(5) Recurrence Relations.

(6) Reciprocal Relations.

(b) The Discrete Case. Polynomials corresponding to Gram's in one variable.

(1) The F.M.G.F. of $G_{r.s.}(x,y)\, w(x,y)$

(2) The Form of $G_{r.s}(x,y)$

(3) A Determinantal Approach.

(4) The Value of $\overset{n}{\underset{0}{S}}\, S\; P^2_{r.s}\,(x,y)\, w(x,y)$

(c) Polynomials corresponding to Charlier's.

(a) The Continuous Case. Polynomials in two variables corresponding to Hermite's in one variable.

$H_r(x)$, the rth Hermite polynomial is defined by

$$H_r(x)\, e^{-\frac{1}{2}x^2} = (-)^r \frac{d^r}{dx^r} e^{-\frac{1}{2}x^2}$$

and

$$H_r(x) = x^r - \frac{r(r-1)}{2}\frac{x^{r-2}}{1!} + \frac{r(r-1)(r-2)(r-3)}{2^2}\frac{x^{r-4}}{2!} - \dots\dots$$

$$= e^{-\frac{1}{2}D^2}x^r \quad \text{where } D = \frac{d}{dx}$$

The orthogonal properties can be written

$$\int_{-\infty}^{\infty} H_r(x)H_s(x)\, e^{-\frac{1}{2}x^2}\, dx \quad = 0 \qquad s \neq 0$$

$$= \sqrt{2\pi}\, r! \qquad r=s$$

There is a recurrence relation, namely

$$H_{r+1}(x) - x\, H_r(x) + r\, H_{r-1}(x) \;=\; 0$$

It is natural to seek an extension of the idea of polynomials in one variable to polynomials in two variables. In some cases the extension leads to little of interest, for example, suppose we try to find the polynomials in x and y orthogonal with respect to $e^{-\frac{1}{2}(x^2+y^2)}$. As will be evident later, these polynomials are merely products such as $H_r(x)H_s(y)$ and all results can be factorised into two independent factors in x and y. This is also true of Tchebychef polynomials where we define the two variable polynomial $T_{r.s}(x,y)$ by

$$\sum_0^n \sum_0^m T_{r.s.}(x,y)\, T_{r's'}(x,y) \;=\; 0 \qquad r'+s' < r+s$$

$$\neq 0 \qquad r'+s' = r+s.$$

in which case $T_{r.s}(x,y)$ has two independent factors, one a Tchebychef polynomial in x and the other in y. To return to Hermite's polynomials, we observe that they arise as an extension of the normal frequency function and are orthogonal with respect to this frequency function. It is suggested from this that the nucleus in the two variable case should be the frequency function in the case of two correlated variates. Hence instead of considering $w(x,y) = e^{-\frac{1}{2}(x^2+y^2)}$ we take

$$w(x,y) = e^{-(x^2-2rxy+y^2)/2(1-r^2)}$$

and look for polynomials $H_{r.s}(x,y)$ defined by

$$\iint_{-\infty}^{\infty} H_{i'k'}(x,y)\, H_{i.k}(x,y)\, e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)}\, dx\, dy$$

$$\begin{aligned} &= 0 \quad i'+k' < i+k \\ &\neq 0 \quad i'+k' = i+k \end{aligned}$$

We shall use Mehler's series for the normal correlation function,

$$w(x,y,r) = \frac{1}{2\pi(1-r^2)^{\frac{1}{2}}}\, e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)}$$

$$= w(x,y,0)\left[1 + rH_1(x)H_1(y) + \frac{r^2}{2!}H_2(x)H_2(y) + \frac{r^3}{3!}H_3(x)H_3(y) + \cdots\right]$$

(Mehler, G. 1866 "Reihentwicklungen nach Laplaceeschen Functionen höherer Ordnung" J. fur Math. lxvl.)

and we shall require integrals of the type

$$\int_{-\infty}^{\infty} H_r(x)\, H_s(x)\, H_p(x)\, e^{-\frac{1}{2}x^2}\, dx$$

a (I)

Starting from $\int_{-\infty}^{\infty} H_k^2(x)\, e^{-\frac{1}{2}x^2}\, dx = \sqrt{2\pi}\, k!$, by integration by parts we find

$$\int_{-\infty}^{\infty} H_k^2(x) H_2(x)\, e^{-\frac{1}{2}x^2}\, dx = 2!\sqrt{2\pi}\, k\, k! \quad \text{since clearly}$$

$$\int_{-\infty}^{\infty} H_k^2(x) H_1(x)\, e^{-\frac{1}{2}x^2}\, dx = 0 \qquad \int_{-\infty}^{\infty} H_k(x) H_{k+1}(x) H_3(x)\, e^{-\frac{1}{2}x^2}\, dx = \frac{3!\,k}{2!\,2!}(k+1)!\sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} H_k^2(x)\, H_4(x)\, e^{-\frac{1}{2}x^2}\, dx = \frac{4!}{2!\,2!}\, k^{(2)} k!$$

and from these we deduce the more general forms

$$\int_{-\infty}^{\infty} H_k^2(x) H_{2s}(x)\, e^{-\frac{1}{2}x^2}\, dx = \frac{(2s)!}{s!\,s!}\, k^{(s)}\, k!\sqrt{2\pi} \quad \ldots\ldots\ldots(\text{I } 1)$$

$$\int_{-\infty}^{\infty} H_k(x) H_{k+1}(x) H_{2s+1}(x)\, e^{-\frac{1}{2}x^2}\, dx = \frac{(2s+1)!}{s!\,(s+1)!}\, k^{(s)}\, (k+1)!\sqrt{2\pi} \quad \ldots\ldots\ldots(\text{I } 2)$$

A direct approach is afforded by the fact that the frequency generating function of $H_k(x)\, e^{-\frac{1}{2}x^2}$ is $a^k e^{\frac{1}{2}a^2}$

i.e. $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_k(x)\, e^{ax-\frac{1}{2}x^2}\, dx = a^k e^{\frac{1}{2}a^2}$

We can differentiate both sides with respect to a provided

$\int_{-\infty}^{\infty} H_k(x)\, x^s\, e^{ax-\frac{1}{2}x^2}\, dx$ converges uniformly

$\int_{-\infty}^{\infty} H_k(x)\, e^{ax-\frac{1}{2}x^2}\, dx$ converges and $H_k(x)\, e^{ax-\frac{1}{2}x^2}$ is

a continuous function of a and x. We have already shown that $\int_{-\infty}^{\infty} H_k(x)\, x^s\, e^{ax-\frac{1}{2}x^2}\, dx$ converges uniformly and hence

$$\frac{d^s}{da^s}\sqrt{\frac{1}{2\Pi}}\int_{-\infty}^{\infty} H_k(x)\, e^{ax-\frac{1}{2}x^2}\,dx = \sqrt{\frac{1}{2\Pi}}\int_{-\infty}^{\infty} H_k(x)\, x^s\, e^{ax-\frac{1}{2}x^2}\,dx$$

$$= \frac{d^s}{da^s}\left[a^k\, e^{\frac{1}{2}a^2}\right]$$

Now since $H_s(x) = x^s - \frac{s^{(2)}}{2\ 1!}x^{s-2} + \frac{s^{(4)}}{2^2\ 2!}x^{s-4} - \ldots\ldots\ldots\ldots$

we have

$$\sqrt{\frac{1}{2\Pi}}\int_{-\infty}^{\infty} H_s(x)H_k(x)e^{ax-\frac{1}{2}x^2}\,dx$$

$$= \left[D_a^s - \frac{s^{(2)}}{2\ 1!}D_a^{s-2} + \frac{s^{(4)}}{2^2\ 2!}D_a^{s-4} - \frac{s^{(6)}}{2^3\ 3!}D_a^{s-6} + \ldots\right]e^{\frac{1}{2}a^2}a^k$$

Now write

$$A_s(x) = x^s + \frac{s^{(2)}x^{s-2}}{2\ 1!} + \frac{s^{(4)}x^{s-4}}{2^2\ 2!} + \frac{s^{(6)}x^{s-6}}{2^3\ 3!} + \ldots\ldots$$

$$= e^{\frac{1}{2}D^2}x^s \quad \text{where } D = \frac{d}{dx}$$

Clearly $e^{-\frac{1}{2}D^2}A_s(x) = x^s$ and $\frac{d}{dx}A_s(x) = s\,A_{s-1}(x)$

i.e.

$$x^s = A_s(x) - \frac{s(s-1)}{2\ 1!}A_{s-2}(x) + \frac{s(s-1)(s-2)(s-3)}{2^2\ 2!}A_{s-4}(x) - \ldots\ldots$$

Hence $\sqrt{\frac{1}{2\Pi}}\int_{-\infty}^{\infty} H_s(x)H_k(x)\, e^{ax-\frac{1}{2}x^2}\,dx$

$$= \left[D_a^s - \frac{s(s-1)}{2\ 1!}D_a^{s-2} + \frac{s(s-1)(s-2)(s-3)}{2^2\ 2!}D_a^{s-4}\ldots\right]e^{\frac{1}{2}a^2}a^k$$

$$= e^{\frac{1}{2}a^2}\Bigg[\left(a^kA_s(a) + ska^{k-1}A_{s-1}(a) + s_{(2)}k^{(2)}a^{k-2}A_{s-2}(a) + \ldots\right)$$

$$-\frac{s^{(2)}}{2\ 1!}\left(a^kA_{s-2} + (s-2)ka^{k-1}A_{s-3} + (s-2)_{(2)}k^{(2)}a^{k-2}A_{s-4} + \ldots\right)$$

$$+\frac{s^{(4)}}{2^2\ 2!}\left(a^kA_{s-4} + (s-4)ka^{k-1}A_{s-5} + (s-4)_{(2)}k^{(2)}a^{k-2}A_{s-6} + \ldots\ldots\right)$$

$$- - - - - - - \Bigg]$$

$$=e^{\frac{1}{2}a^2}\Big[a^k\Big(A_s(a)-\frac{s^{(2)}}{2\ 1!}A_{s-2}(a)+\frac{s^{(4)}}{2^2\ 2!}A_{s-4}(a)-\ldots\ldots\Big)$$

$$+ska^{k-1}\Big(A_{s-1}(a)-\frac{(s-1)^{(2)}}{2\ 1!}A_{s-3}+\frac{(s-1)^{(4)}}{2^2\ 2!}A_{s-5}(a)-\ldots\Big)$$

$$+s^{(2)}k^{(2)}a^{k-2}\Big(A_{s-2}(a)-\frac{(s-2)^{(2)}}{2\ 1!}A_{s-4}(a)+\frac{(s-2)^{(4)}}{2^2\ 2!}A_{s-6}(a)-\ldots\Big)$$

$$+\ldots\ldots\ldots\ldots\ldots\Big]$$ a terminating series since k is integral.

$$=e^{\frac{1}{2}a^2}\Big[a^{k+s}+sk\,a^{k+s-2}+\frac{s^{(2)}k^{(2)}}{2!}a^{k+s-4}+\frac{s^{(3)}k^{(3)}}{3!}a^{k+s-6}+\ldots+\frac{s^{(s)}k^{(s)}}{s!}a^{k-s}\Big]\qquad k\geqslant s$$

Hence
$$\int_{-\infty}^{\infty}H_s(x)H_k(x)\,e^{-\frac{1}{2}x^2}\,dx = 0 \qquad k\neq s$$
$$= \sqrt{2\pi}\ k! \qquad k=s$$

Again
$$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}H_k^2(x)\,e^{ax-\frac{1}{2}x^2}\,dx=e^{\frac{1}{2}a^2}\Big[a^{2k}+\frac{k^2a^{2k-2}}{1!}+\frac{(k^{(2)})^2}{2!}a^{2k-4}+\ldots+\frac{(k!)^2}{k!}\Big]$$

$$= U(a)\quad\text{say.}$$

We determine $\int_{-\infty}^{\infty}H_k^2(x)H_{2s}(x)e^{-\frac{1}{2}x^2}\,dx$ from $\int_{-\infty}^{\infty}H_k^2(x)e^{ax-\frac{1}{2}x^2}\,dx$

as we did $\int_{-\infty}^{\infty}H_k(x)H_s(x)e^{-\frac{1}{2}x^2}\,dx$ from $\int_{-\infty}^{\infty}H_k(x)\,e^{ax-\frac{1}{2}x^2}\,dx$

For $H_{2s}(x)=x^{2s}-\frac{(2s)^{(2)}}{2\ 1!}x^{2s-2}+\frac{(2s)^{(4)}}{2^2\ 2!}x^{2s-4}-\frac{(2s)^{(6)}}{2^3\ 3!}x^{2s-6}+\ldots\ldots$

Hence $\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} H_k^2(x)H_{2s}(x)\, e^{-\frac{1}{2}x^2}\, dx$

$$= \left[\left(D_a^{2s} - \frac{(2s)^{(2)}}{2\ 1!} D_a^{2s-2} + \frac{(2s)^{(4)}}{2^2\ 2!} D_a^{2s-4} - \frac{(2s)^{(6)}}{2^3\ 3!} D^{2s-6} + \ldots\ldots \right) U(a) \right]_{a=0}$$

$$= \Big[\ V(a)\ \Big]_{a=0}$$

Expanding each term of $V(a)$ and picking out the non-vanishing terms we have

$V(0) =$

$$(2s)!\frac{k^2(k-1)^2\ldots(s+1)^2}{(k-s)!} + (2s-2)!(2s)^{(2)}\frac{k^2(k-1)^2\ldots\ldots s^2}{(k-s+1)!} + \ldots.$$

$$\frac{-2s(2s-1)}{2\ 1!}\left((2s-2)!\frac{k^2(k-1)^2\ldots s^2}{(k-s+1)!} + (2s-4)!(2s-2)^{(2)}\frac{k^2(k-1)^2\ldots(s-1)^2}{(k-s+2)!} + \ldots . \right)$$

$$+ \frac{2s(2s-1)(2s-2)(2s-3)}{2^2\ \ 2!}\left((2s-4)!\frac{k^2(k-1)^2\ldots(s-1)^2}{(k-s+2)!} + \ldots\ldots\ldots \right)$$

$$= (2s)!\ \frac{k^2(k-1)^2\ldots\ldots(s+1)^2}{(k-s)!} \qquad = \frac{(2s)!}{s!\ \ s!}\ k!\ k^{(s)} \qquad k \geq s$$

Hence $\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} H_k^2(x)\ H_{2s}(x)\ e^{-\frac{1}{2}x^2}\ dx \ = \frac{(2s)!}{s!\ s!}\ k!\ k^{(s)}$

and similarly

$$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} H_k(x)H_{k+1}(x)\ H_{2s+1}(x)e^{-\frac{1}{2}x^2}\ dx = \frac{(2s+1)!(k+1)!\ k^{(s)}}{s!(s+1)!}$$

which establishes results (I1) and (I2) on page 102.

We give two further results which are sometimes useful :-

$$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} H_k(x)H_{k-(2r+1)}(x)\ H_{2s+1}(x)e^{-\frac{1}{2}x^2}\ dx=\frac{(2s+1)!\,(k-2r-1)!\,k^{(s+r+1)}}{(s-r)!\,(s+r+1)!}$$

where $k \geqslant 2r+1$, $k \geqslant s+r+1$ $\quad s \geqslant r$ ......(I3)

$$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} H_k(x)H_{k-2r}(x)\ H_{2s}(x)\ e^{-\frac{1}{2}x^2}\ dx = \frac{(2s)!\quad (k-2r)!}{(s-r)!\,(s+r)!}\,k^{(s+r)}$$

where $k \geqslant 2r$, $k \geqslant s+r$, $s \geqslant r$ . ...(I4)

The Form of $H_{i.k.}(x,y)$

If we assume

$H_{i.k}(x,y)=a_{i.k}x^iy^k+ a_{i.k-1}x^iy^{k-1}+ a_{i-1.k}x^{i-1}y^k+$ ...... terms of lower degree,

then the coefficients $a_{i.k}$ are determined by

$$\iint_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)}\ H_{i'.k'}(x,y)\ H_{i.k}(x,y)\ dx\ dy = 0,\ i'+k' < i+k$$

We find

$$H_{i.k}(x,y) = H_i(x)H_k(y)-rikH_{i-1}(x)H_{k-1}(y)+\frac{r^2}{2!}i^{(2)}k^{(2)}H_{i-2}(x)H_{k-2}(y)$$
$$-\frac{r^3}{3!}i^{(3)}k^{(3)}H_{i-3}(x)H_{k-3}(y) + \dots\dots\dots\dots$$

the truth of which can be demonstrated by direct substitution

in $\displaystyle\iint_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)}\ H_{i.k}(x,y)\ H_{i'.k'}(x,y)\ dx\ dy$

and the use of Mehler's series for $w(x,y,r)$. A simpler proof is given later .

<u>(a) ii</u>

<u>The Value of</u> <u>(A)</u> $\displaystyle\iint_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)} H^2_{i.k}(x,y)\ dx\ dy$

<u>(B)</u> $\displaystyle\iint_{-\infty}^{\infty} H_{i.k}(x,y)H_{i'.k'}(x,y)e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)}\ dx\ dy$

$$(A) \iint_{-\infty}^{\infty} H^2_{i.k}(x,y)\ e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)}\ dx\ dy =$$

$$=2\pi(1-r^2)^{\frac{1}{2}}\Big[H_i(x)H_k(y)\ H_i(x)H_k(y)-rikH_{i-1}(x)H_{k-1}(y) + \frac{r^2}{2!}i^{(2)}k^{(2)}H_{i-2}(x)H_{k-2}(y) - \ldots\ldots\Big]\Big[1+rH_1(x)H_1(y) + \frac{r^2}{2!}H_2(x)H_2(y) + \ldots\ldots\Big]\ e^{-\frac{1}{2}(x^2+y^2)}\ dx\ dy$$

$$=2\pi(1-r^2)^{\frac{1}{2}}\left\{ \frac{i!k!}{0!}\left(1+ik2!r^2+\frac{i^{(2)}k^{(2)}4!r^4}{2!2!2!2!}+\frac{i^{(3)}k^{(3)}6!r^6}{3!3!3!3!}+\ldots+\frac{i^{(k)}k!(2k)!r^{2k}}{k!k!k!k!}\right)\right.$$

$$-\frac{i!k!}{1!}\left(ikr^2+\frac{i^{(2)}k^{(2)}3!r^4}{1!1!2!2!}+\frac{i^{(3)}k^{(3)}5!r^6}{2!2!3!3!}+\ldots+\frac{i^{(k)}k!(2k-1)!r^{2k}}{(k-1)!(k-1)!k!k!}\right)$$

$$+\frac{i!k!}{2!}\left(\frac{i^{(2)}k^{(2)}2!r^4}{0!0!2!2!}+\frac{i^{(3)}k^{(3)}4!r^6}{1!1!3!3!}+\ldots+\frac{i^{(k)}k!(2k-2)!\ r^{2k}}{(k-2)!(k-2)!k!k!}\right)$$

$$-\frac{i!k!}{3!}\left(\frac{i^{(3)}k^{(3)}3!r^6}{0!0!3!3!}+\frac{i^{(4)}k^{(4)}5!r^8}{1!1!4!4!}+\ldots+\frac{i^{(k)}k!\ (2k-3)!\ r^{2k}}{(k-3)!(k-3)!k!k!}\right)$$

$$\left.+\ \ldots\ldots\ldots\ldots\ldots\ldots \right\}$$

$$=2\pi(1-r^2)^{\frac{1}{2}}\ i!k!\left(1+\frac{ikr^2}{1!1!}+\frac{i^{(2)}k^{(2)}r^4}{2!2!}+\frac{i^{(3)}k^{(3)}r^6}{3!3!}+\frac{i^{(4)}k^{(4)}r^8}{4!4!}+\ldots\frac{i^{(k)}k!r^{2k}}{k!k!}\right)$$

$i \geq k$

using results (I1),(I2),(I3) and (I4) .

By a similar method, the result (B) is

$$\iint_{-\infty}^{\infty} H_{i.k}(x,y)\, H_{i'.k'}(x,y)\quad e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)}\quad dx\, dy$$

$$=2\pi(1-r^2)^{\frac{1}{2}}\frac{i!k'!r^{k'-k}}{(k'-k)!}\left[1+\frac{i'kr^2}{(i-i'+1)1!}+\frac{i'^{(2)}k^{(2)}r^4}{(i-i'+2)^{(2)}2!}+\frac{i'^{(3)}k^{(3)}r^6}{(i-i'+3)^{(3)}3!}+\cdots\cdots+\frac{i'^{(k)}k!\,r^{2k}}{(i-i'+k)^{(k)}k!}\right]$$

where $i \geqslant i'$ $\quad$ $i+k=i'+k'$
$k' \geqslant k$

(a)iii A SYMBOLIC FORM FOR $H_{i.k}(x,y)$

We have seen that

$$H_{i.k}(x,y)= H_i(x)H_k(y)- rikH_{i-1}(x)H_{k-1}(y) +\frac{r^2}{2!}i^{(2)}k^{(2)}H_{i-2}(x)H_{k-2}(y)$$

$$-\frac{r^3}{3!}i^{(3)}k^{(3)}H_{i-3}(x)H_{k-3}(y) + \cdots\cdots\cdots$$

Now
$$H_i(x) = e^{-\frac{1}{2}D_x^2}\, x^i$$

Hence
$$H_{i.k}(x,y)= e^{-\frac{1}{2}(D_x^2+ D_y^2)}\, x^iy^k-rikx^{i-1}y^{k-1} +\frac{r^2}{2!}i^{(2)}k^{(2)}x^{i-2}y^{k-2}$$

$$-\frac{r^3}{3!}i^{(3)}k^{(3)}x^{i-3}y^{k-3}+ \cdots\cdots$$

$$= e^{-\frac{1}{2}(D_x^2+D_y^2)}\; e^{-rD_xD_y}\; x^i\, y^k$$

$$=\underline{e^{-\frac{1}{2}(D_x^2+ 2rD_xD_y + D_y^2)}\; x^i\, y^k}$$

If r=0, so that the variables are independent

$$H_{i.k}(x,y) = e^{-\frac{1}{2}(D_x^2 + D_y^2)} x^i y^k$$

$$= H_i(x)\, H_k(y)$$

i.e. the polynomials developed by $e^{-\frac{1}{2}(x^2+y^2)}$ are merely products of H(x) and H(y) .

(a) iv

AN APPROACH BY GENERATING FUNCTIONS AND THEIR DERIVATIVES.

The generating function of $H_r(x)\, e^{-\frac{1}{2}x^2}$ where

$$H_r(x) = x^r - \frac{r^{(2)} x^{r-2}}{2\ 1!} + \frac{r^{(4)} x^{r-4}}{2^2\ 2!} - \frac{r^{(6)} x^{r-6}}{2^3\ 3!} + \ldots\ldots.$$

is $a^r e^{\frac{1}{2}a^2}$.

i.e. $$\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} H_r(x)\, e^{ax-\frac{1}{2}x^2}\, dx = a^r e^{\frac{1}{2}a^2}$$

Now suppose f(x) is any polynomial in x . Then since

$$\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} x^s\, e^{ax-\frac{1}{2}x^2}\, dx = D_a^s\, e^{\frac{1}{2}a^2}$$

the differentiation being justified since $\int_{-\infty}^{\infty} x^s e^{ax-\frac{1}{2}x^2}\, dx$ converges uniformly and $\int_{-\infty}^{\infty} e^{ax-\frac{1}{2}x^2}\, dx$ converges,

$$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ax-\frac{1}{2}x^2}\,dx = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} e^{ax-\frac{1}{2}x^2}\,dx\left[f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots\right]$$

$$=\left[f(0)+f'(0)D_a+\frac{f''(0)}{2!}D_a^2+\ldots\ldots\right]e^{\frac{1}{2}a^2}$$

$$=e^{\frac{1}{2}a^2}\left[\frac{f(0)}{0!}+A_1(a)\frac{f'(0)}{1!}+A_2(a)\frac{f''(0)}{2!}+A_3(a)\frac{f'''(0)}{3!}+\ldots\right]$$

$$=e^{\frac{1}{2}a^2}\,e^{\frac{1}{2}D_a^2}\,f(a)$$

i.e. $$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ax-\frac{1}{2}x^2}\,dx = e^{\frac{1}{2}a^2}\,e^{\frac{1}{2}D_a^2}\,f(a) \quad \ldots\ldots\ldots(15)$$

where $D_a = \frac{d}{da}$

The extension of this result to two variables gives a ready result for Hermite polynomials in two variables, but it will be clearer to show the derivation of these polynomials in one variable first. In the above result, take

$$f(a) = e^{-\frac{1}{2}D_a^2}\,a^r$$

which makes f(a) a polynomial in 'a' of degree r.

Then $$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}\left(e^{-\frac{1}{2}D_x^2}\,x^r\right)e^{ax-\frac{1}{2}x^2}\,dx = e^{\frac{1}{2}a^2}\,e^{\frac{1}{2}D_a^2}\,e^{-\frac{1}{2}D_a^2}\,a^r$$

$$= a^r\,e^{\frac{1}{2}a^2}$$

Hence $e^{-\frac{1}{2}D^2}x^r$ is a polynomial of degree r which satisfies

$$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}\left(e^{-\frac{1}{2}D_x^2}\,x^r\right)x^s\,e^{ax-\frac{1}{2}x^2}\,dx = \frac{d^s}{da^s}\left(a^r\,e^{\frac{1}{2}a^2}\right)$$

and so $\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}\left(e^{-\frac{1}{2}D^2}x^r\right)x^s e^{-\frac{1}{2}x^2}\,dx = 0 \quad s\neq r$

$= r! \quad s=r$

i.e. $e^{-\frac{1}{2}D^2}x^r$ is identical with Hermite's polynomial.

In two variables, if $f(x,y)$ is any polynomial in x and y, then

$$\iint_{-\infty}^{\infty} f(x,y)e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)}\, e^{ax+by}\,dx\,dy$$
$$=2\pi(1-r^2)^{\frac{1}{2}}\ \text{Exp.}\ \tfrac{1}{2}(a^2+2rab+b^2)\ \text{Exp.}\frac{1}{2(1-r^2)}(D_a^2-2rD_aD_b+D_b^2)f(A,B)$$

where $A= a+rb \quad B= b+ra$ and Exp. means 'exponential'.

<u>Proof.</u> We know that the frequency generating function of the normal correlation function $w(x,y,r)$ is $\text{Exp.}\frac{1}{2}(a^2+2rab+b^2)$

i.e.

$$\frac{1}{2\pi(1-r^2)^{\frac{1}{2}}}\iint_{-\infty}^{\infty} e^{ax+by}\ \text{Exp.}\ -\frac{(x^2-2rxy+y^2)}{2(1-r^2)}\,dx\,dy$$
$$= \text{Exp.}\tfrac{1}{2}(a^2+2rab+b^2)$$

Differentiate both sides i times with respect to a
and k " " " " b

Then

$$\frac{1}{2\pi(1-r^2)^{\frac{1}{2}}}\iint_{-\infty}^{\infty} w(x,y,r)x^iy^k e^{ax+by}\,dx\,dy= \frac{d^i}{da^i}\frac{d^k}{db^k}e^{\frac{1}{2}(a^2+2rab+b^2)}$$

the justification of which depends on the single variable case which has been demonstrated.

(Note that $\frac{d}{da}$ and $\frac{d}{db}$ represent partial differentiation in this case.)

Now $\frac{du}{da} = \frac{du}{dA} + p\frac{du}{dB}$ and $\frac{du}{db} = p\,\frac{du}{dA} + \frac{du}{dB}$

and so $\frac{d^i}{da^i}\frac{d^k}{db^k}\, e^{\frac{1}{2}(a^2+2pab+b^2)} = \left(\frac{d}{dA} + p\frac{d}{dB}\right)^i \left(p\frac{d}{dA} + \frac{d}{dB}\right)^k e^{\frac{(A^2-2pAB+B^2)}{2(1-p^2)}}$

$= \left(p\frac{d}{dA} + \frac{d}{dB}\right)^k A_i(A)\; e^{\frac{1}{2}(A^2-2pAB+B^2)/(1-p^2)}$

$= e^{\frac{1}{2}(A^2-2pAB+B^2)/(1-p^2)}\; A_i(A)A_k(B) + ikpA_{i-1}(A)A_{k-1}(B) + \ldots\ldots\ldots$

$= e^{\frac{1}{2}(a^2+2pab+b^2)}\; e^{\frac{1}{2}(D_a^2-2pD_aD_b+D_b^2)/(1-p^2)}\; (a+pb)^i\,(b+pa)^k$

Hence if $f(x,y)$ is any polynomial in x,y then

$$\frac{1}{2\pi(1-r^2)^{\frac{1}{2}}}\iint_{-\infty}^{\infty} w(x,y,r)\, f(x,y)\, e^{ax+by}\, dx\, dy$$

$$= e^{\frac{1}{2}(a^2+2rab+b^2)}\, w^{-1}\left(\frac{d}{da}, \frac{d}{db}, r\right) f(a+rb, b+ra)$$

where $w(x,y,r) = e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)}$

Now $f(x,y)$ is an arbitrary polynomial . Hence take

$$f(a+rb, ra+b) = e^{-\frac{1}{2}(D_a^2-2rD_aD_b+D_b^2)/(1-r^2)}\, (a+rb)^i (b+ra)^k$$

i.e. $f(x,y) = f_{i.k}(x,y) = e^{-\frac{1}{2}(D_x^2+2rD_xD_y+D_y^2)}\, x^i\, y^k$

and then

$$\iint_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)}\, f_{i.k}(x,y)\, x^{i'} y^{k'}\, dx\, dy =$$

$$2\,\pi(1-r^2)^{\frac{1}{2}}\left[\frac{d^{i'+k'}}{da^{i'}\,db^{k'}}\, (a+rb)^i (b+ra)^k\, e^{\frac{1}{2}(a^2+2rab+b^2)}\right]_{\substack{a=0\\b=0}}$$

Hence

$$\int\!\!\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)} f_{i.k}(x,y)x^{i'}y^{k'}dx\,dy = 0 \qquad i'+k' \quad i+k$$

$$\neq 0 \qquad i'+k'=i+k$$

We have thus proved that $f_{i.k}(x,y) = \left[\mathrm{Exp.}-\frac{1}{2}(D_x^2+2rD_xD_y+D_y^2)\right]x^iy^k$ represent orthogonal polynomials in two variables with respect to $w(x,y,r)$. Clearly

$$f_{i.k}(x,y) = \left(e^{-rD_xD_y}\right)\left(e^{-\frac{1}{2}D_x^2}\,x^i\right)\left(e^{-\frac{1}{2}D_y^2}\,y^k\right)$$

$$= e^{-rD_xD_y}\left(H_i(x)\,H_k(y)\right)$$

$$= H_i(x)H_k(y) - rikH_{i-1}(x)H_{k-1}(y) + \frac{r^2}{2!}i^{(2)}k^{(2)}H_{i-2}(x)H_{k-2}(y) - \ldots$$

which is identical with the form given on page 105.

Expanding $\dfrac{d^{i'+k'}}{da^{i'}db^{k'}}(a+rb)^i\,(b+ra)^k\,e^{\frac{1}{2}(a^2+2rab+b^2)}$

and putting $a=b=0$, we have $\dfrac{1}{2\pi(1-r^2)^{\frac{1}{2}}}\displaystyle\int\!\!\int_{-\infty}^{\infty} H_{i.k}\,H_{i'.k'}\,w(x,y,r)dx\,dy$

$$= \left[\frac{d^{i'+k'}}{da^{i'}db^{k'}}\left(i_{(i')}a^{i'}(rb)^{i-i'} + i_{(i'-1)}a^{i'-1}(rb)^{i-i'+1} + \ldots\right)\times\right.$$

$$\left(b^{i'+k'-i} + (i'+k'-1)b^{i'+k'-i-1}(ra) + (i'+k'-1)_{(2)}b^{i'+k'-i-2}(ra)^2 + \ldots\right)$$

$$\left. e^{\frac{1}{2}(a^2+2rab+b^2)}\right]_{a=b=0}$$

$$= i_{(i')}r^{i-i'}i'!k'! + i_{(i'-1)}r^{i-i'+1}(i'+k'-i)ri'!k'!$$

$$+ i_{(i'-2)}r^{i-i'+2}(i'+k'-i)_{(2)}r^2i'!k'! + \ldots\ldots\ldots$$

$$= i'!\,k'!\,r^{i-i'}i_{(i')}\left[1 + \frac{i'kr^2}{(i-i'+1)1!} + \frac{i'^{(2)}k^{(2)}r^4}{(i-i'+2)^{(2)}\,2!} + \frac{i'^{(3)}k^{(3)}r^6}{(i-i'+3)^{(3)}3!} + \ldots\right]$$

Hence we have the result

$$\int\int_{-\infty}^{\infty} H_{i.k}(x,y)H_{i'.k'}(x,y)\, e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}}\, dx\, dy = 0 \qquad i'+k' < i+k$$

$$=2\pi(1-r^2)^{\frac{1}{2}} i'!k'!r^{i-i'}\, i_{(i')}\left[1+\frac{i'kr^2}{(i-i'+1)1!}+\frac{i'^{(2)}k^{(2)}r^4}{(i-i'+2)^{(2)}2!}+\frac{i'^{(3)}k^{(3)}r^6}{(i-i'+3)^{(3)}3!} + \ldots\ldots\right] \quad \text{if } i'+k'=i+k,\ i'\leq i$$

It is easily seen that when $i'=i$ and $k'=k$

$$\int\int_{-\infty}^{\infty} H^2_{i.k}(x,y)\, e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)}\, dx\, dy$$

$$=2\pi(1-r^2)^{\frac{1}{2}} i!k!\left[1+\frac{ikr^2}{1!1!}+\frac{i^{(2)}k^{(2)}r^4}{2!2!}+\frac{i^{(3)}k^{(3)}r^6}{3!3!}+\ldots\ldots\ldots\right]$$

It has been shown that $\displaystyle\int\int_{-\infty}^{\infty} f(x,y)\, w(x,y,r)\, e^{ax+by}\, dx\, dy$

$$=2\pi(1-r^2)\, e^{\frac{1}{2}(a^2+2rab+b^2)}\, e^{\frac{1}{2}(D_a^2-2rD_aD_b+D_b^2)/(1-r^2)} f(a+rb,b+ra)$$

Take

$$f(a+rb,ra+b) = e^{-\frac{1}{2}(D_a^2-2rD_aD_b+D_b^2)/(1-r^2)}\, a^i b^k$$

so that

$$f(x,y) = e^{-\frac{1}{2}(D_x^2+2rD_xD_y+D_y^2)}\, \frac{(x-ry)^i(y-rx)^k}{(1-r^2)^{i+k}}$$

then $\displaystyle\int\int_{-\infty}^{\infty} f_{i.k}(x,y)\, f_{i'.k'}(x,y)\, w(x,y,r)\, dx\, dy$

$$= 0 \qquad i'+k' < i+k$$

$$=\left[\frac{d^{i'+k'}}{da^{i'}\,db^{k'}}\, a^i b^k\, e^{\frac{1}{2}(a^2+2rab+b^2)}\right]_{a=b=0}$$

$$=i!k! \qquad \text{when } i=i' \text{ and } k=k'$$

Hence if Hermite polynomials in two variables are defined by

$$A_{i.k}(x,y) = \left[\mathrm{Exp.}-\tfrac{1}{2}(D_x^2+2rD_xD_y+D_y^2)\right](x-ry)^i\,(y-rx)^k \qquad \text{then}$$

$$\iint_{-\infty}^{\infty} A_{i.k}(x,y)\, A_{i'.k'}(x,y)\, w(x,y,r)\, dx\, dy = 0 \quad \text{unless } \begin{matrix} i=i' \\ k=k' \end{matrix}$$

$$= 2\pi(1-r^2)^{i+k}\, i!k!$$

which is to be compared with

$$\iint_{-\infty}^{\infty} H_{i.k}(x,y)\, H_{i'k'}(x,y)\, dx\, dy \quad = 0 \qquad i'+k' \quad i+k$$

$$\neq 0 \qquad i'+k' = i+k$$

Another Form for $A_{i.k}(x,y)$

We have page 111

$$\frac{d^{i+k}}{da^i db^k}\, e^{\frac{1}{2}(a^2+2rab+b^2)} = e^{\frac{1}{2}(a^2+2rab+b^2)}\, e^{\frac{\frac{1}{2}(D_a^2-2rD_aD_b+D_b^2)}{(1-r^2)}}(a+rb)^i(b+ra)^k$$

Put $a = \frac{ix}{(1-r^2)^{\frac{1}{2}}}$ and $b = \frac{iy}{(1-r^2)^{\frac{1}{2}}}$ and $-r$ for r and we have (where $i=\sqrt{-1}$)

$$\frac{d^{i+k}}{dx^i dy^k}\, e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}} = e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}}\, e^{-\frac{1}{2}(D_x^2+2rD_xD_y+D_y^2)}\frac{(x-ry)^i(y-rx)^k}{(1-r^2)^{i+k}}$$

Hence

$$(-)^{i+k} A_{i.k}(x,y) = (1-r^2)^{i+k}\, e^{\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}}\, D_x^i D_y^k\, e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}}$$

or

$$e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}}\, A_{i.k}(x,y) = (r^2-1)^{i+k}\left(\frac{d}{dx}\right)^i\left(\frac{d}{dy}\right)^k e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}}$$

which is to be compared with

$$e^{-\frac{1}{2}x^2}\, H_r(x) = (-)^r \left(\frac{d}{dx}\right)^r e^{-\frac{1}{2}x^2}$$

in the one variable case.

We proceed now to the consideration of recurrence relations satisfied by these polynomials in two variables.

(a) v

We have $H_{i.k}(x,y) = e^{-\frac{1}{2}(D_x^2-2rD_xD_y+D_y^2)} x^i y^k$

Hence $\frac{dH}{dx}{}_{i.k} = iH_{i-1.k}$ $\quad \frac{dH}{dy}{}_{i.k} = kH_{i.k-1}$

Further

$$(D_x+pD_y)^i(pD_x+D_y)^k e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}} = (-)^{i+k} e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}} H_{i.k}$$

Operate with $D_x + pD_y$ and then

$$(D_x+pD_y)^{i+1}(pD_x+D_y)^k e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}} = (-)^{i+k}\left[\left(\frac{ry-x+r^2x-ry}{1-r^2}\right) H_{i.k} + iH_{i-1.k} + rkH_{i.k-1}\right] e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}}$$

i.e.

$$(-)^{i+k+1}H_{i+1.k} = (-)^{i+k}\ -xH_{i.k} + i\,H_{i-1.k} + rkH_{i.k-1}$$

Hence

$$H_{i+1.k}(x,y) - xH_{i.k}(x,y) + iH_{i-1.k}(x,y) + rkH_{i.k-1}(x,y) = 0$$

Similarly

$$H_{i,k+1}(x,y) - yH_{i.k}(x,y) + kH_{i.k-1}(x,y) + riH_{i-1.k}(x,y) = 0$$

and combining the above relations,

$$r(i-k)\, H_{i.k}(x,y) + xH_{i.k+1}(x,y) - yH_{i+1.k}(x,y) + kH_{i+1.k-1}(x,y) - iH_{i-1.k+1}(x,y) = 0$$

Again $A_{i.k}(x,y) = e^{-\frac{1}{2}(D_x^2+2rD_xD_y+D_y^2)}(x-ry)^i\,(y-rx)^k$

Hence

$$\frac{dA}{dx}{}_{i.k} = iA_{i-1.k} - rkA_{i.k-1} \qquad \frac{dA}{dy}{}_{i.k} = -riA_{i-1.k} + kA_{i.k-1}$$

Further

$$(-)^{i+k}\, e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}}\; A_{i.k}=(1-r^2)^{i+k}\, D_x^i\, D_y^k\, e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}}$$

Differentiate with respect to x and then

$$(-)^{i+k}\left[-\frac{(x-ry)}{1-r^2}\, A_{i.k} + D_x A_{i.k}\right] e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)}$$

$$= (1-r^2)^{i+k}\, D_x^{i+1} D_y^k\; e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}}$$

$$=(1-r^2)^{-1}(-)^{i+k+1}\, e^{-\frac{1}{2}(x^2-2rxy+y^2)/(1-r^2)}\, A_{i+1.k}$$

Hence

$$A_{i+1.k}(x,y) - (x-ry)A_{i.k}(x,y) +(1-r^2)(iA_{i-1.k} -rkA_{i.k-1}) = 0$$

Similarly

$$A_{i.k+1}(x,y) -(y-rx)A_{i.k}(x,y) +(1-r^2)(kA_{i.k-1} -riA_{i-1.k}) = 0$$

and eliminating $A_{i.k}$ from these two results we have

$$(y-rx)A_{i+1.k} +(ry-x)A_{i.k+1} +(1-r^2)^2(iyA_{i-1.k} -kxA_{i.k-1}) = 0$$

<u>(a) Vl</u> <u>RECIPROCAL RELATIONS.</u>

Since $H_{i.k}(x,y) = e^{-\frac{1}{2}(D_x^2 +2rD_xD_y +D_y^2)}\; x^i\, y^k$

and $\frac{dH}{dx}{}_{i.k} = iH_{i-1.k}$ $\frac{dH}{dy}{}_{i.k} = kH_{i.k-1}$

it follows that

$$x^i\, y^k = e^{\frac{1}{2}(D_x^2+2rD_xD_y+D_y^2)}\; H_{i.k}(x,y)$$

Expanding the right hand side in terms of D_x and D_y

$$x^iy^k = H_{i.k} + \tfrac{1}{2}\left(i^{(2)}H_{i-2.k} + 2rikH_{i-1.k-1} + k^{(2)}H_{i.k-2}\right)$$
$$+ \frac{1}{2^2\,2!}\left(i^{(4)}H_{i-4.k} + 4r^2 i^{(2)}k^{(2)}H_{i-2.k-2} + k^{(4)}H_{i.k-4} + \right.$$
$$\left. 4ri^{(3)}kH_{i-3.k} + 4rik^{(3)}H_{i.k-3} + 2i^{(2)}k^{(2)}H_{i-2.k-2}\right) + \cdots\cdots$$

Again $$e^{\frac{1}{2}(D_x^2+D_y^2)}H_{i.k} = e^{-rD_xD_y}\,x^i\,y^k$$

i.e.

$$H_{i.k} + \tfrac{1}{2}\left(i^{(2)}H_{i-2.k} + k^{(2)}H_{i.k-2}\right) + \frac{1}{2^2\,2!}\left(i^{(4)}H_{i-4.k} + 2i^{(2)}k^{(2)}H_{i-2.k-2} + k^{(4)}H_{i.k-4}\right) + \cdots\cdots$$

$$= x^iy^k - rikx^{i-1}y^{k-1} + \frac{r^2}{2!}i^{(2)}k^{(2)}x^{i-2}y^{k-2} - \frac{r^3}{3!}i^{(3)}k^{(3)}x^{i-3}y^{k-3} + \cdots\cdots\cdots$$

The polynomial form in x,y is seen to be very similar to the form for $H_{i.k}$ in terms of $H(x)$, $H(y)$, namely

$$H_{i.k} = H_{ix}H_{ky} - rikH_{i-1.x}H_{k-1.y} + \frac{r^2}{2!}i^{(2)}k^{(2)}H_{i-2.x}H_{k-2.y} - \cdots\cdots\cdots$$

Lastly $$e^{rD_xD_y}H_{i.k} = e^{-\frac{1}{2}(D_x^2+D_y^2)}\,x^i\,y^k$$

i.e.

$$H_{i.k} + rikH_{i-1.k-1} + \frac{r^2}{2!}i^{(2)}k^{(2)}H_{i-2.k-2} + \frac{r^3}{3!}i^{(3)}k^{(3)}H_{i-3.k-3} + \cdots\cdots$$

$$= x^iy^k - \frac{x^{i-2}y^{k-2}}{2\;1!}\left(i^{(2)}y^2 + k^{(2)}x^2\right) + \frac{x^{i-4}y^{k-4}}{2^2\;2!}\left(i^{(4)}y^4 + 2i^{(2)}k^{(2)}x^2y^2 + k^{(4)}x^4\right)$$
$$- \cdots\cdots\cdots\cdots\cdots$$

A similar set of relations exist for $A_{i.k}(x,y)$,for

$$A_{i.k.}(x,y) = e^{-\frac{1}{2}(D_x^2 + 2rD_xD_y + D_y^2)} (x-ry)^i (y-rx)^k$$

Hence

$$(x-ry)^i (y-rx)^k = A_{i.k} + \frac{(1-r^2)}{2\ 1!} \left(i^{(2)} A_{i-2.k} - 2ikrA_{i-1.k-1} + k^{(2)} A_{i.k-2}\right)$$
$$+ \frac{(1-r^2)^2}{2^2\ 2!} \left(i^{(4)} A_{i-4.k} + 4r^2 i^{(2)} k^{(2)} A_{i-2.k-2} + k^{(4)} A_{i.k-4} - \ldots\ldots\right)$$

THE RELATION BETWEEN $A_{i.k}(x,y)$ and $H_{i.k}(x,y)$

Put $\quad x-ry = (1-r^2)^{\frac{1}{2}} X$

$\quad\quad\ y-rx = -(1-r^2)^{\frac{1}{2}} Y$

and then

$$D_x = (D_X + rD_Y) / (1-r^2)^{\frac{1}{2}}$$

$$D_y = (-rD_X - D_Y) / (1-r^2)^{\frac{1}{2}}$$

Hence

$$A_{i.k}(x,y) = e^{-\frac{1}{2}(D_x^2 + 2rD_xD_y + D_y^2)} (x-ry)^i (y-rx)^k$$
$$= e^{-\frac{1}{2}(D_X^2 + 2rD_XD_Y + D_Y^2)} X^i Y^k (1-r^2)^{(i+k)/2} (-)^k$$

and so

$$A_{i.k}(x,y) = (-)^k (1-r^2)^{(i+k)/2} H_{i.k}(X,Y)$$
$$= (-)^k (1-r^2)^{(i+k)/2} H_{i.k}\left(\frac{x-ry}{(1-r^2)^{\frac{1}{2}}}, \frac{rx-y}{(1-r^2)^{\frac{1}{2}}}\right)$$

or

$$A_{i.k}\left(\frac{X-rY}{(1-r^2)^{\frac{1}{2}}}, \frac{rX-Y}{(1-r^2)^{\frac{1}{2}}}\right) = (-)^k (1-r^2)^{\frac{(i+k)}{2}} H_{i.k}(X,Y)$$

(b) THE DISCRETE CASE

(1) The Factorial Moment Generating Function of $G_{r.s}(x,y)$

We have seen that the $r^{th.}$ Gram polynomial $G_r(x)$ can be defined by:-

$$G_r(x) = (pq)^r \; p^{-x} \; q^{x-n} \; \frac{d^r}{dp^r} \; p^x q^{n-x}$$

An extension to two variables is suggested, namely

$\frac{d^r}{dp^r} \frac{d^s}{dp'^s} \; p^x \, p'^{\,y} \, (1-p-p')^{n-x-y}$ for this is the coefficient

of $t^x u^y$ in $(p_{00} + pt + p'u)^n$ corresponding to the 'fourfold table' of probability

	B	$\bar{B}$	
A	p_{11}	p_{10}	p
$\bar{A}$	p_{01}	p_{00}	q
	p'	q'	1

where $p_{11} = 0$

For $\quad (1+pa+p'b)^n = SS \, (1+a)^x \, (1+b)^y \; w(x,y)$

where $\quad w(x,y) = \frac{n^{(x+y)}}{x! \; y!} \, p^x \, p'^y \, p_{oo}^{\,n-x-y}$

Hence differentiating r times with respect to p

and s " " " " p'

$$S\,S\,(1+a)^x(1+b)^y \frac{d^{r+s}}{dp^r dp'^s} w(x,y) = a^r b^s n^{(r+s)} (1+pa+p'b)^{n-r-s}$$

Now set

$$p^x p'^y (1-p-p')^{n-x-y} G_{r.s}(x,y) = p^r p'^s p_{00}^{r+s} \frac{d^{r+s}}{dp^r dp'^s} p^x p'^y (1-p-p')^{n-x-y}$$

It is clear that $G_{r.s}(x,y)$ is a polynomial in x and y of degree r+s. Further

$$SSx^{(i)}y^{(k)}w(x,y)G_{r.s}(x,y) = p^r p'^s p_{oo}^{r+s} n^{(r+s)} \frac{d^{i+k}}{da^i db^k} a^r b^s (1+pa+p'b)^{n-r-s}$$

when a=b=0

$$= 0 \qquad i+k < r+s$$

$$= p^r p'^s p_{oo}^{r+s} n^{(r+s)} r!\, s!$$

when $i=r$, $k=s$.

Clearly $G_{r.s}(x,y)$ are orthogonal polynomials with respect to $w(x,y)$ and the F.M.G.F. of $G_{r.s}(x,y)\, w(x,y)$ is

$$n^{(r+s)} p^r p'^s (1-p-p')^{r+s} a^r b^s (1+pa+pb)^{n-r-s}$$

where $w(x,y)\, G_{r.s}(x,y) = p^r p'^s (1-p-p')^{r+s} \frac{d^{r+s}}{dp^r dp'^s} w(x,y)$

---

(11) <u>The Form of $G_{r.s}(x,y)$</u>

An alternative definition of $G_{r.s}(x,y)$ is now given. It is easily proved that if G(a) is the factorial moment generating function of g(x), then $a^r G(a)$ is the factorial moment generating function of $(-)^r \Delta^r g(x-r)$, where $\Delta u_x = u_{x+1} - u_x$

The extension to two dimensions gives

$a^r b^s\, G(a,b)$ as the F.M.G.F. of $(-)^{r+s} \Delta_x^r \Delta_y^s\, g(x-r, y-s)$

Apply this theorem to $G(a,b) = (1+pa+p'b)^n$. Then

$$a^r b^s \frac{d^{r+s}}{da^r db^s} (1+pa+p'b)^n \text{ generates } (-)^{r+s} \Delta_x^r \Delta_y^s\, x^{(r)} y^{(s)} g(x,y)$$

$$\text{where } g(x,y) = w(x,y) = \frac{n^{(x+y)}}{x!\ y!} p^x p'^y (1-p-p')^{n-x-y}$$

i.e.

$$a^r b^s p^r p'^s\, n^{(r+s)} (1+pa+p'b)^{n-r-s} = SS(1+a)^x (1+b)^y (-)^{r+s} \Delta_x^r \Delta_y^s x^{(r)} y^{(s)} w(x,y)$$

But

$$a^r b^s p^r p'^s (1-p-p')^{r+s} (1+pa+p'b)^{n-r-s} = SS(1+a)^x (1+b)^y\, G_{r.s}(x,y)\, w(x,y)$$

from paragraph (1) page 120. Hence

$$G_{r.s}(x,y)\, w(x,y) = (-)^{r+s}\, p_{oo}^{r+s} \Delta_x^r \Delta_y^s\, x^{(r)} y^{(s)} w(x,y)$$

which is analogous to

$$e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}} A_{i.k}(x,y) = (-)^{i+k} (1-r^2)^{i+k} D_x^i D_y^k\, e^{-\frac{1}{2}\frac{(x^2-2rxy+y^2)}{(1-r^2)}}$$

in two variables or to

$$G_r(x)\, w(x) = (-q)^r \Delta_x^r\, x^{(r)}\, w(x)$$

in one variable.

We now find the expression for $G_{r.s}(x,y)$ in terms of x and y.

$G_{r.s}(x,y)$ in terms of x and y . Since

$$p^x p'^y p_{oo}^{n-x-y}\, G_{r.s}(x,y) = p^r p'^s p_{oo}^{r+s} \frac{d^{r+s}}{dp^r dp'^s} p^x p'^y (1-p-p')^{n-x-y}$$

$$= p^r p'^s p_{oo}^{r+s} \frac{d^s}{dp'^s} \Big\{ x^{(r)} p^{x-r} p_{oo}^{n-x-y} - r x^{(r-1)} p^{x-r+1} (n-x-y) p_{oo}^{n-x-y-1}$$

$$+ r_{(2)} x^{(r-2)} p^{x-r+2} (n-x-y)^{(2)} p_{oo}^{n-x-y-2} \ldots\ldots \ldots \Big\} p'^y$$

$$= p_{oo}^{r+s} p'^s \Big\{ x^{(r)} p^x \Big(y^{(s)} p'^{y-s} p_{oo}^{n-x-y} - s y^{(s-1)} p'^{y-s+1} (n-x-y) p_{oo}^{n-x-y-1}$$

$$+ s_{(2)} y^{(s-2)} p'^{y-s+2} (n-x-y)^{(2)} p_{oo}^{n-x-y-2} \ldots\ldots\ldots\ldots \Big)$$

$$- r x^{(r-1)} (n-x-y) p^{x+1} \Big(y^{(s)} p_{oo}^{n-x-y-1} p'^{y-s} - s y^{(s-1)} p'^{y-s+1} (n-x-y-1) p_{oo}$$

$$+ s_{(2)} y^{(s-2)} p'^{y-s+2} (n-x-y-1)^{(2)} p_{oo}^{n-x-y-3} + \ldots\ldots \Big)$$

$$+ r_{(2)} x^{(r-2)} (n-x-y)^{(2)} p^{x+2} \Big(y^{(s)} p'^{y-s} p_{oo}^{n-x-y-2} - s y^{(s-1)} p'^{y-s+1} (n-x-y-2) \times p_{oo}^{n-x-y-3} \ldots \Big) \Big\}$$

then

$$G_{r.s}(x,y) = p_{oo}^{r+s} \Big[\Big(x^{(r)} y^{(s)} - s p' \frac{(n-x-y)}{p_{oo}} x^{(r)} y^{(s-1)} + s_{(2)} x^{(r)} p'^2 (n-x-y)^{(2)} \ldots\ldots \Big)$$

$$- r x^{(r-1)} (n-x-y) p \Big(\frac{y^{(s)}}{p_{oo}} - s y^{(s-1)} p' \frac{(n-x-y-1)}{p_{oo}^2} + s_{(2)} y^{(s-2)} p'^2 \frac{(n-x-y-1)^{(2)}}{p_{oo}^3} \Big)$$

$$+ r_{(2)} x^{(r-2)} (n-x-y)^{(2)} p^2 \Big(\frac{y^{(s)}}{p_{oo}^2} - s y^{(s-1)} p' \frac{(n-x-y-2)}{p_{oo}^3} + \ldots\ldots\ldots \Big) \Big]$$

For example $G_{1.0}(x,y) = p_{oo} x - p(n-x-y)$

$G_{0.1}(x,y) = p_{oo} y - p'(n-x-y)$

$G_{2.0}(x,y) = p_{oo}^2 x^{(2)} - 2pp_{oo} x(n-x-y) + p^2 (n-x-y)^{(2)}$

$$G_{11}(x,y)=p_{oo}^2xy-p'p_{oo}x(n-x-y)-pp_{oo}y(n-x-y)+pp'(n-x-y)^{(2)}$$

$$G_{o.2}(x,y)=p_{oo}^2y^{(2)}-2p'p_{oo}y(n-x-y)+p'^2(n-x-y)^{(2)}$$

(b)111 A Determinantal Approach.

We consider a few simple cases of polynomials orthogonal with respect to $\frac{n^{(x+y)}}{x!y!}p^xp'^yp_{oo}^{n-x-y}$ and build up a general expression which is proved by means of its factorial moment generating function. The following results for Gram's polynomials in one variable are required:-

$$G_1^2(x) = G_2(x) + (1-2p)G_1(x) + npq$$

$$G_1(x)G_2(x)= G_3(x)+2(1-2p)G_2(x)+ 2pq(n-1)\ G_1(x)$$

$$G_2^2(x)=G_4(x)+ 4(1-2p)G_3(x)+\left(4npq-16pq+2\right) G_2(x) +4pq(1-2p)(n-1)G_1(x) +2n(n-1)(pq)^2$$

These and other results required are special cases of the following:-

$$G_r(x)G_1(x) = G_{r+1}(x) + \left(r(1-2p)G_r(x) + rpq(n-r+1)\right)G_{r-1}(x)$$

$$G_r(x)G_2(x) = G_{r+2}(x) +2r(1-2p)G_{r+1}(x) +\left(2rpq(n-r)+r^{(2)}(1-2p)^2\right)G_r(x) +2r^{(2)}pq(n-r+1)(1-2p)G_{r-1}(x) +r^{(2)}(pq)^2(n-r+2)(n-r+1)G_{r-2}(x)$$

$$G_r(x)G_3(x)=G_{r+3}(x)+3r(1-2p)G_{r+2}(x)+3r\left[(n-r-1)pq+(r-1)(1-2p)^2\right]G_{r+1}$$
$$+\left[(1-2p)^3r^{(3)}+6pq(1-2p)(n-r)r^{(2)}\right]G_r(x)$$
$$+3(n-r+1)pqr^{(2)}\left[pq(n-r)+(r-2)(1-2p)^2\right]G_{r-1}(x)$$
$$+3(pq)^2(1-2p)(n-r+2)^{(2)}r^{(3)}G_{r-2}(x)$$
$$+r^{(3)}(pq)^3(n-r+3)^{(3)}G_{r-3}(x)$$

$$G_r(x)G_4(x)=G_{r+4}+4r(1-2p)G_{r+3}+\left[4rpq(n-r+2)+6r^{(2)}(1-2p)^2\right]G_{r+2}$$
$$+\left[12r^{(2)}(n-r-1)pq+4r^{(3)}(1-2p)^2\right](1-2p)G_{r+1}$$
$$+6pq(n-r)r^{(2)}\left[2(r-2)(1-2p)^2+pq(n-r-1)\right]G_r(x)$$
$$+4r^{(3)}(n-r+1)(1-2p)pq\left[3(n-r)pq+(r-3)(1-2p)^2\right]G_{r-1}(x)$$
$$+2r^{(3)}(pq)^2(n-r+2)^{(2)}\left[2pq(n-r)+3(r-3)(1-2p)^2\right]G_{r-2}(x)$$
$$+4(pq)^3(n-r+3)^{(3)}(1-2p)r^{(4)}G_{r-3}(x)+r^{(4)}(pq)^4(n-r+4)^{(4)}G_{r-4}$$

and finally if $r \geqslant s$

$$G_r(x)G_s(x)=G_{r+s}(x)+sr(1-2p)G_{r+s-1}(x)+\left[s_{(2)}r^{(2)}(1-2p)^2+spqr(n-r-s+2)\right]G_{r+s-2}(x)$$
$$+\left[s_{(3)}r^{(3)}(1-2p)^3+s^{(2)}r^{(2)}pq(n-r-s+3)(1-2p)\right]G_{r+s-3}(x)$$
$$+\left[s_{(4)}r^{(4)}(1-2p)^4+3s_{(3)}r^{(3)}pq(n-r-s+4)(1-2p)^2+s_{(2)}r^{(2)}(pq)^2(n-r-s+4)^{(2)}\right]\times G_{r+s-4}(x)$$
$$+\ \cdots\cdots\cdots$$
$$+\Big[s_{(k)}r^{(k)}(1-2p)^k+srpq(n-r-s+k)(s-1)_{(k-2)}(r-1)^{(k-2)}(1-2p)$$
$$+s_{(2)}r^{(2)}p^2q^2(n-r-s+k)^{(2)}(s-2)_{(k-4)}(r-2)^{(k-4)}(1-2p)^{k-4}$$
$$+s_{(3)}r^{(3)}p^3q^3(n-r-s+k)^{(3)}(s-3)_{(k-6)}(r-3)^{(k-6)}(1-2p)^{k-6}$$
$$+\qquad\Big]G_{r+s-k}(x)+\cdots\cdots\cdots$$

from which any product such as

$$\mathop{S}_{0}^{n} G_r(x)G_s(x)G_t(x)w(x) \text{ can be written down.}$$

The result

$$\frac{n^{(x+y)}}{x!y!} p^x p'^y (1-p-p')^{n-x-y} = w(x,p)w(y,p') \left\{ 1 + \frac{dG_1(x)G_1(y)}{n^1 pp' q^1 q'} + \frac{d^2}{2n^{(2)}(pp'qq')^2} G_2(x)G_2(y) + \frac{d^3}{3!\, n^{(3)} (pp'qq')^3} G_3(x) G_3(y) + \ldots \right\}$$

where $\begin{cases} d = -pp' \\ w(x,p) = n_{(x)} p^x q^{n-x} \end{cases}$, is required.

Take $g_{oo}(x,y) = 1$ and $g_{1.0}(x,y) = a + b\, G_1(x)$

where $S\, S\, g_{1.0}(x,y)\, w(x,y) = 0$.

i.e. $a = 0$ and so $g_{1.o} = G_1(x)$.

Similarly $g_{0.1}(x,y) = G_1(y)$ and $g_{2.0}(x,y) = G_2(x)$

Assume $g_{1.1}(x,y) = aG_1(x)G_1(y) + bG_1(x) + cG_1(y) + d$

with the conditions to determine a,b,c and d as follows:-

$SS\, g_{11}(x,y)\, w(x,y) = 0 \quad SS\, g_{11}(x,y)\, w(x,y)\, G_1(x) = 0$

$SS\, g_{11}(x,y)\, w(x,y)\, G_1(y) = 0$

Using the expression for g_{11} above in addition to these three equations we can eliminate a,b,c and d and obtain

$$g_{11}(x,y) = \begin{vmatrix} 1 & G_1(x) & G_1(y) & G_1(x)G_1(y) \\ 1 & 0 & 0 & -npp' \\ 0 & npq & -npp' & -npp'(1-2p) \\ 0 & -npp' & np'q' & -npp'(1-2p') \end{vmatrix}$$

and after division by a constant,

$$g_{ii}(x,y) = G_1(x)G_1(y) + pG_1(y) + p'G_1(x) + npp'$$

Again $SS\, g_{11}^2(x,y)\, w(x,y) = SS\, g_{11}(x,y)\, G_1(x)G_1(y) + pG_1(y) +$

$$p'G_1(x) + npp' \quad w(x,y)$$

$$= SSg_{11}(x,y)G_1(x)\, G_1(y)\, w(x,y) \quad \text{since}$$

$$SSg_{11}(x,y)G_1(y)w(x,y) = SSg_{11}(x,y)G_1(x)w(x,y) = SSg_{11}(x,y)w(x,y) = 0$$

Hence

$$SSg_{11}^2(x,y)\, w(x,y) = n^{(2)}pp'(pp'+qq')$$

Similarly if $g_{21}(x,y) = aG_2(x) + bG_2(x)G_1(y) + cG_1(x)G_1(y) + dG_2(y)$

$$+ eG_1(x) + fG_1(y) + h$$

with the conditions

$SSg_{21}(x,y)w(x,y) = SSg_{21}(x,y)G_1(x)w(x,y) = SSg_{21}(x,y)G_1(y)w(x,y)$

$= SSg_{21}(x,y)G_2(x)w(x,y) = SSg_{21}(x,y)G_1(x)G_1(y)w(x,y)$

$= SSg_{21}(x,y)G_2(y)w(x,y) = 0$, then $g_{21}(x,y)$ has the form

$$g_{21}(x,y) =$$

$$\begin{vmatrix}
1 & G_1(x) & G_1(y) & G_2(x) & G_1(x)G_1(y) & G_2(y) & G_2(x)G_1(y) \\
1 & 0 & 0 & 0 & -npp' & 0 & 0 \\
0 & npq & -npp' & 0 & -npp'(1-2p) & 0 & -2pqn^{(2)}pp' \\
0 & -npp' & np'q' & 0 & -npp'(1-2p') & 0 & 2n^{(2)}(pp')^2 \\
0 & 0 & 0 & 2n^{(2)}p^2q^2 & -2n^{(2)}pqpp' & 2n^{(2)}(pp')^2 & -4pq(1-2p)n^{(2)}pp' \\
-npp' & -npp'(1-2p) & -npp'(1-2p') & -2n^{(2)}pp'pq & \begin{matrix} 2n^{(2)}p^2p'^2 \\ -(1-2p)(1-2p')npp' \\ +n^2pp'qq' \end{matrix} & -2n^{(2)}pp'^2q' & \begin{matrix} 4(1-2p)n^{(2)}p^2p'^2 \\ -2pq(1-2p') \\ \times n^{(2)}pp' \end{matrix} \\
0 & 0 & 0 & 2n^{(2)}(pp')^2 & -2n^{(2)}p'q'pp' & 2n^{(2)}p'^2q'^2 & \begin{matrix} 4n^{(2)}(1-2p') \\ \times p^2p'^2 \end{matrix}
\end{vmatrix}$$

After some simplification we have :-

$$g_{21}(x,y) = G_2(x)G_1(y) + 2p'G_2(x) + 2pG_1(x)G_1(y) + 2npp'G_1(x) + 2p^2G_1(y) + 2np^2p'$$

Similarly

$$g_{12}(x,y) = G_1(x)G_2(y) + 2pG_2(y) + 2p'G_1(x)G_1(y) + 2npp'G_1(y) + 2p'^2G_1(x) + 2npp'^2$$

The first few polynomials are now set out :-

$$g_{00}=1 \quad g_{10}=G_1(x) \quad g_{01}=G_1(y) \quad g_{20}=G_2(x) \quad g_{02}=G_2(y)$$

$$g_{11}=G_1(x)G_1(y)+pG_1(y)+p'G_1(x)+npp'$$

$$g_{30}=G_3(x)$$

$$g_{21}=G_2(x)G_1(y)+2p'G_2(x)+2pG_1(x)G_1(y)+2npp'G_1(x)+2p^2G_1(y)+2np^2p'$$

$$g_{12}=G_1(x)G_2(y)+2pG_2(y)+2p'G_1(x)G_1(y)+2npp'G_1(y)+2p'^2G_1(x)+2npp'^2$$

$$g_{03}=G_3(y)$$

$$g_{40}=G_4(x)$$

$$g_{31}=G_3(x)G_1(y)+3p'G_3(x)+3pG_2(x)G_1(y)+3npp'G_2(x)+6p^2G_1(x)G_1(y)$$
$$+6np^2p'G_1(x)+6p^3G_1(y)+6np^3p'$$

$$g_{22}=G_2(x)G_2(y)+4p'G_2(x)G_1(y)+4pG_1(x)G_2(y)+6p'^2G_2(x)$$
$$+4(n+1)pp'G_1(x)G_1(y)+6p^2G_2(y)+8npp'(p'G_1(x)+pG_1(y))$$
$$+2p^2p'^2n(n+3)$$

$$g_{13}=G_1(x)G_3(y)+3pG_3(y)+3p'G_1(x)G_2(y)+3npp'G_2(y)+6np'^2pG_1(y)$$
$$+6p'^3G_1(x)+6npp'^3$$

$$g_{04}=G_4(y)$$

$SSg^2_{oo}w = 1$ $\quad SSg^2_{10}\ w = npq$ $\quad SSg^2_{20}\ w = 2!(pq)^2\ n^{(2)}$

$SSg^2_{01}\ w = np'q'$ $\quad SSg^2_{11}\ w = pp'(pp'+qq')n^{(2)}$

$SSg^2_{02}\ w = 2!(p'q')^2\ n^{(2)}$

$SS\ g^2_{30}\ w = 3!(pq)^3 n^{(3)}$ $\quad SSg^2_{21}\ w = 2!n^{(3)}p^2p'q(2pp'+qq')$

$SS\ g^2_{03}\ w = 3!(p'q')^3 n^{(3)}$ $\quad SSg^2_{12}\ w = 2!n^{(3)}pp'^2q'(2pp'+qq')$

$SSg^2_{40}\ w = 4!n^{(4)}(pq)^4$ $\quad SSg^2_{31}\ w = 3!1!n^{(4)}p^3p'q^2(3pp'+qq')$

$SSg^2_{04}\ w = 4!n^{(4)}(p'q')^4$ $\quad SSg^2_{22}w = 2!2!n^{(4)}p^2p'^2(p^2p'^2+4pp'qq'+q^2q'^2)$

$SSg^2_{13}w = 1!3!n^{(4)}pp'^3q'^2(3pp'+qq')$

and by using

$G_r(x,n-s) = G_r(x) + rspG_{r-1}(x) + r_{(2)}(s+1)^{(2)}p^2G_{r-2}(x) + \ldots\ldots\ldots$

(proved on page 97) , we easily verify that

$g_{11} = G_1(x,n-1)G_1(y,n-1)+(n-1)pp'$

$g_{21} = G_2(x,n-1)G_1(y,n-2)+2(n-2)pp'G_1(x,n-1)$

$g_{12} = G_1(x,n-2)G_2(y,n-1)+2(n-2)pp'G_1(y,n-1)$

$g_{22} = G_2(x,n-2)G_2(y,n-2)+4(n-3)pp'G_1(x,n-2)G_1(y,n-2)+2(n-2)^{(2)}p^2p'^2$

$g_{31} = G_3(x,n-1)G_1(y,n-3)+3(n-3)pp'G_2(x,n-1)$

$g_{13} = G_1(x,n-3)G_3(y,n-1)+3(n-3)pp'G_2(y,n-1)$

We observe that each of these expressions is of the form

$$g_{rs} = G_r(x,n-s)G_s(y,n-r)+(n-r-s+1)pp'rsG_{r-1}(x,n-s)G_{s-1}(y,n-r)$$
$$+(n-r-s+2)_{(2)}s^{(2)}r^{(2)}(pp')^2G_{r-2}(x,n-s)G_{s-2}(y,n-r)+\ldots$$

but clearly to attempt a proof by determinantal methods as used here would not be easy. We consider an approach by use of the frequency generating function of $w(x,y)$.

We have seen that

$$SS\, t^x u^y\, w(x,y) = (p_{oo}+pt+p'u)^n \quad \text{where } w(x,y) = \frac{n^{(x+y)}}{x!\,y!}\, p^x p'^y p_{oo}^{n-x-y}$$

$$\text{and } p_{oo}+p+p'=1$$

Now $SS\, x^{(r)} y^{(s)} w(x,y)\, t^x u^y = n^{(r+s)} p^r p'^s t^r u^s (p_{oo}+pt+p'u)^{n-r-s}$

by differentiating the identity above r times for t and s times for u.

Now $G_r(x) = (1+p\Delta)^{-n+r-1} x^{(r)}$ and it is natural to look for an extension in the form

$(1+p\Delta_x+p'\Delta_y)^{-n+r+s-1} x^{(r)} y^{(s)}$. Let us find the frequency generating function of $w(x,y)(1+p\Delta_x+p'\Delta_y)^{-n+r+s-1} x^{(r)} y^{(s)}$

It is

$$SSt^x u^y w(x,y)(1+p\Delta_x+p'\Delta_y)^{-n+r+s-1} x^{(r)} y^{(s)} .$$

$$=SSt^x u^y w(x,y)\left[x^{(r)} y^{(s)} -(n-r-s+1)\left(prx^{(r-1)} y^{(s)} +p'sx^{(r)} y^{(s-1)}\right) + \ldots\ldots\right]$$

$$=n^{(r+s)} p^r p'^s t^r u^s (p_{oo}+pt+p'u)^{n-r-s} -(n-r-s+1)\Big[prn^{(r+s-1)} p^{r-1} p'^s \times$$

$$t^{r-1} u^s (p_{oo}+pt+p'u)^{n-r-s+1} +sp'n^{(r+s-1)} p^r p'^{s-1} t^r u^{s-1} (p_{oo}+pt+p'u)^{n-r-s+1}\Big]$$

$$+(n-r-s+2)_{(2)}\Big[r^{(2)} p^2 n^{(r+s-2)} p^{r-2} p'^s t^{r-2} u^s +2pp'rsn^{(r+s-2)} p^{r-1} p'^{s-1}$$

$$+p'^2 s^{(2)} n^{(r+s-2)} p^r p'^{s-2} t^r u^{s-2}\Big] (p_{oo}+pt+p'u)^{n-r-s+2}$$

$+ \ldots\ldots\ldots\ldots$.

$$=n^{(r+s)}p^r p'^s(p_{oo}+pt+p'u)^{n-r-s}\Big\{t^r u^s-(rt^{r-1}u^s+st^r u^{s-1})C$$

$$+\frac{1}{2!}(r^{(2)}t^{r-2}u^s+2rst^{r-1}u^{s-1}+s^{(2)}t^r u^{s-2})C^2$$

$$-\frac{1}{3!}(r^{(3)}t^{r-3}u^s+3r^{(2)}st^{r-2}u^{s-1}+3rs^{(2)}t^{r-1}u^{s-2}+s^{(3)}t^r u^{s-3})C^3$$

$$+\ \ldots\ldots\ldots\ldots\ldots\Big\}$$

$$=n^{(r+s)}p^r p'^s(p_{oo}+pt+p'u)^{n-r-s}\Big\{u^s\Big(t^r-rt^{r-1}C+r_{(2)}t^{r-2}C^2+\ldots\Big)$$

$$-su^{s-1}\Big(t^r-rt^{r-1}C+\ldots\ldots\ldots\Big)C\quad+\ldots\ldots\Big\}$$

$$=n^{(r+s)}p^r p'^s C^{n-r-s}\Big\{\Big[u^s-su^{s-1}C+s_{(2)}u^{s-2}C^2-s_{(3)}u^{s-3}C^3+\ldots\Big]\Big[t^r-$$

$$rt^{r-1}C+r_{(2)}t^{r-2}C^2-r_{(3)}t^{r-3}C^3+\ldots\ldots\ldots\Big]\Big\}$$

$$=n^{(r+s)}p^r p'^s\, C^{n-r-s}\,(u-C)^s(t-C)^r \quad \text{where } C=p_{oo}+pt+p'u$$

$$=n^{(r+s)}p^r p'^s(p_{oo}+pt+p'u)^{n-r-s}\,(q'u-pt-p_{oo})^s(qt-p'u-p_{oo})^r$$

Hence

$$\sum_0^n S\ w(x,y)\ t^x u^y\ (1+p\Delta_x+p'\Delta_y)^{-n+r+s-1}\ x^{(r)}\ y^{(s)}$$

$$=n^{(r+s)}p^r p'^s(p_{oo}+pt+p'u)^{n-r-s}(qt-p'u-p_{oo})^r(q'u-pt-p_{oo})^s$$

Putting

$$qt-p'u-p_{oo}=A$$

$$q'u-pt-p_{oo}=B$$

$$p_{oo}+pt+p'u=C$$

we have

$$SSt^x u^y w(x,y)(1+p\Delta_x+p'\Delta_y)^{-n+r+s-1} x^{(r)} y^{(s)} = n^{(r+s)} p^r p'^s A^r B^s C^{n-r-s}$$

and differentiating both sides i times with respect to t and k times with respect to u, putting t=u=1

$$SSw(x,y)x^{(i)} y^{(k)} (1+p\Delta_x+p'\Delta_y)^{-n+r+s-1} x^{(r)} y^{(s)}$$

$$= \left[\frac{d^i}{dt^i} \frac{d^k}{du^k} \; n^{(r+s)} p^r p'^s A^r B^s C^{n-r-s} \right]_{t=u=1}$$

$$= 0 \qquad\qquad i+k < r+s$$

$$\neq 0 \qquad\qquad i+k = r+s$$

since A=B=0 , C=1 when t=u=1

<u>Hence $(1+p\Delta_x+p'\Delta_y)^{-n+r+s-1} x^{(r)} y^{(s)}$ are orthogonal polynomials with respect to $w(x,y) = \dfrac{n^{(x+y)} p^x p'^y (1-p-p')^{n-x-y}}{x!y!}$</u>

<u>$P_{r.s}(x,y) = (1+p\Delta_x+p'\Delta_y)^{-n+r+s-1} x^{(r)} y^{(s)}$ in terms of x, y.</u>

By expanding $(1+p\Delta_x+p'\Delta_y)^{-n+r+s-1}$ we have

$$P_{r.s}(x,y) = \Big[1 - (n-r-s+1)(p\Delta_x+p'\Delta_y) + (n-r-s+2)_{(2)}(p\Delta_x+p'\Delta_y)^2$$
$$-(n-r-s+3)_{(3)}(p\Delta_x+p'\Delta_y)^3 \;\ldots\ldots\ldots\ldots \Big] x^{(r)} y^{(s)}$$

$$= x^{(r)} y^{(s)} - (n-r-s+1)(prx^{(r-1)} y^{(s)} + sp'x^{(r)} y^{(s-1)})$$
$$+ (n-r-s+2)_{(2)} \Big(p^2 r^{(2)} x^{(r-2)} y^{(s)} + 2pp'rsx^{(r-1)} y^{(s-1)}$$
$$+ p'^2 s^{(2)} x^{(r)} y^{(s-2)} \Big) + - - - -$$

Again $(1+pt+p'u)^{-n+r+s-1} = \left[(1+pt)(1+p'u)-pp'tu\right]^{-n+r+s-1}$

and $(1+p\Delta_x)^{-n+r-1} x^{(r)} = G_r(x)$

Hence

$$(1+p\Delta_x+p'\Delta_y)^{-n+r+s-1} x^{(r)} y^{(s)} = \left[(1+p\Delta_x)(1+p'\Delta_y)-pp'\Delta_x\Delta_y\right]^{-n+r+s-1} x^{(r)} y^{(s)}$$

$$=P_{r.s}(x,y)$$

$$=G_r(x,n-s)G_s(y,n-r)+(n-r-s+1)rspp'G_{r-1}(x,n-s)G_{s-1}(y,n-r)$$

$$+(n-r-s+2)_{(2)}p^2p'^2r^{(2)}s^{(2)}G_{r-2}(x,n-s)G_{s-2}(y,n-r)+\ldots\ldots$$

i.e.

$$P_{r.s}(x,y)=G_{rx}G_{sy}+(n-r-s+1)pp'rsG_{r-1x}G_{s-1y}+(n-r-s+2)_{(2)}(pp')^2\times$$

$$r^{(2)}s^{(2)}G_{r-2x}G_{s-2y}+\ldots\ldots$$

where $G_{rx}=G_r(x,n-s)$ and $G_{sy}=G_s(y,n-r)$

Using the relation

$$G_r(x,n-s)=G_r(x)+rspG_{r-1}(x)+r_{(2)}(s+1)^{(2)}p^2G_{r-2}(x)+\ldots\ldots\ldots\ldots$$

established in Chapter 2 page 97, we have $P_{r.s}(x,y)=$

$$\left(G_r(x)+rspG_{r-1}(x)+r_{(2)}(s+1)^{(2)}p^2G_{r-2}(x)+\cdots\right)\left(G_s(y)+rsp'G_{s-1}(y)+\ldots\right)$$

$$+(n-r-s+1)pp'rs\left(G_{r-1}(x)+(r-1)spG_{r-2}(x)+\ldots\right)\left(G_{s-1}(y)+(s-1)rp'G_{s-2}(y)+\right)$$

$$+(n-r-s+2)_{(2)}(pp')^2r^{(2)}s^{(2)}\left(G_{r-2}(x)+(r-2)spG_{r-3}(x)+\cdots\right)\left(G_{s-2}(y)+(s-2)rp'G_{s-3}(y)+\cdots\right)$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

For example, taking r=2 and s=1,

$$P_{21}(x,y)=(G_2(x)+2pG_1(x)+2p^2)(G_1(y)+2p') +2.1(n-2)pp'(G_1(x)+p)$$

$$=G_2(x)G_1(y)+2p'G_2(x)+2pG_1(x)G_1(y)+2npp'G_1(x)$$

$$+2p^2G_1(y) +2np^2p'$$

as obtained for $g_{21}(x,y)$ by determinantal methods on page 127.

$x^{(r)}y^{(s)}$ in terms of $P_{r.s}(x,y)$

We have $\Delta_x P_{r.s}(x,y)= \Delta_x(1+p\Delta_x+p'\Delta_y)^{-n+r+s-1}\, x^{(r)}\, y^{(s)}$

$$=(1+p\Delta_x+p'\Delta_y)^{-n+r+s-1}\Delta_x x^{(r)}\, y^{(s)}$$

$$=(1+p\Delta_x+p'\Delta_y)^{-n+r+s-1}\, rx^{(r-1)}\, y^{(s)}$$

$$= r\, P_{r-1.s}(x,y,n-1)$$

Similarly,

$$\Delta_y P_{r.s}(x,y)= sP_{r.s-1}(x,y,n-1)$$

Hence $x^{(r)}y^{(s)} \quad =(1+p\Delta_x+p'\Delta_y)^{n-r-s+1}P_{r.s}(x,y)$

$$=\Big[1+(n-r-s+1)(p\Delta_x+p'\Delta_y)+(n-r-s+1)_{(2)}(p\Delta_x+p'\Delta_y)^2 + \dots\dots\dots \Big]P_{r.s}(x,y)$$

$$=P_{r.s}+(n-r-s+1)\Big[prP_{r-1.s.n-1}+p'sP_{r.s-1.n-1}\Big]$$

$$+(n-r-s+1)_{(2)}\Big[p^2r^{(2)}P_{r-2.s.n-2}+2pp'rsP_{r-1.s-1.n-1}+p'^2s^{(2)}P_{r.s-2.n-2}\Big]$$

$+ \dots\dots\dots\dots\dots$

Hence $P_{r.s} + (n-r-s+1)\left(prP_{r-1.s.n-1} + p'sP_{r.s-1.n-1}\right)$

$+(n-r-s+1)_{(2)}\left(P_{r-2.s.n-2}\,p^2 r^{(2)} + P_{r-1.s-1.n-2}\,2pp'rs + P_{r.s-2.n-2}\,p'^2 s^{(2)}\right)$

$+(n-r-s+1)_{(3)}\left(p^3 r^{(3)} P_{r-3.s.n-3} + 3p^2p' r^{(2)} sP_{r-2.s-1.n-3} + 3pp'^2 rs^{(2)} P_{r-1.s-2.n-3} + p'^3 s^{(3)} P_{r.s-3.n-3}\right)$

$+ \ldots\ldots\ldots\ldots\ldots\ldots$

$= x^{(r)} y^{(s)}$

a terminating series in $P_{r.s}(x,y,n)$

(b) IV THE VALUE OF $S\,S\,P^2_{r.s}(x,y)\,w(x,y)$

We have seen that

$$\overset{n}{\underset{o}{S}}\,S\,w(x,y)\,P_{r.s}(x,y)\,t^x u^y = n^{(r+s)} p^r p'^s A^r B^s C^{n-r-s}$$

where $A = qt-p'u-p_{oo}$ $B = q'u-pt-p_{oo}$ $C = pt+p'u+p_{oo}$

Putting $r-s$ for r, and $s+k$ for s, differentiating r times with respect to t and k times with respect to u, we have

$$SSx^{(r)}y^{(k)}P_{r-s.s+k}(x,y)w(x,y) = n^{(r+k)}p^{r-s}p'^{s+k}\left(\frac{d^{r+k}}{dt^r du^k}A^{r-s}B^{s+k}C^{n-r-k}\right)_{t=u=1}$$

$$=n^{(r+k)} p^{r-s} p'^{s+k}\left[\frac{d^r}{dt^r}\left(C^{n-k-r}\left\{ (r-s)^{(k)}(-p')^k B^{s+k}A^{r-s-k} \right.\right.\right.$$

$$+k(r-s)^{(k-1)}(-p')^{k-1}(s+k)B^{s+k-1}q'A^{r-s-k+1}$$

$$\left.\left.\left. +k_{(2)}(r-s)^{(k-2)}(-p')^{k-2}(s+k)^{(2)}B^{s+k-2}q'^2A^{r-s-k+2}+\dots\right\}+\dots\right)\right]_{t=1}$$

$$=n^{(r+k)}p^{r-s}p'^{s+k}\Big[(r-s)^{(k)}(-p')^k r_{(s+k)}(s+k)!(r-s-k)!(-p)^{s+k}q^{r-s-k}$$

$$+k(r-s)^{(k-1)}(-p')^{k-1}q'(s+k)r_{(s+k-1)}(s+k-1)!(r-s-k+1)!(-p)^{s+k-1}q^{r-s-k+1}$$

$$+\dots\dots\dots\Big]$$

$$=n^{(r+k)}p^rp'^{s+k}q^{r-s-k}r!k!(-)^s\Big[(r-s)_{(k)}(pp')^k+$$

$$(r-s)_{(k-1)}(s+k)(pp')^{k-1}qq'+(r-s)_{(k-2)}(s+k)_{(2)}(pp')^{k-2}(qq')^2$$

$$+(r-s)_{(k-3)}(s+k)_{(3)}(pp')^{k-3}(qq')^3+\dots\dots+(s+k)_{(k)}(qq')^k\Big]$$

Hence we have the two results:- $r \geqslant s$

$$SS\, P_{rk}(x,y)\, P_{r-s.s+k}(x,y)\, w(x,y) =$$

$$n^{(r+k)}p^rp'^{s+k}q^{r-s-k}r!\,k!\,(-)^s\Big[(r-s)_{(k)}(pp')^k+$$

$$(r-s)_{(k-1)}(s+k)(pp')^{k-1}(qq')+(r-s)_{(k-2)}(s+k)_{(2)}(pp')^{k-2}(qq')^2$$

$$+\dots\dots\dots\dots+(s+k)_{(k)}(qq')^k\Big]$$

$$SS\, P^2_{r.s}(x,y)\, w(x,y) = n^{(r+s)}p^r p'^s q^{r-s} r!s!\Big[r_{(s)}(pp')^s+$$

$$sr_{(s-1)}(pp')^{s-1}qq'+s_{(2)}r_{(s-2)}(pp')^{s-2}(qq')^2+\dots\Big]$$

(c) POLYNOMIALS CORRESPONDING TO CHARLIER'S

We have taken the nucleus of the Gram polynomials in two variables to be that generated by

$$(p_{11}ut+p_{10}t + p_{01}u + p_{00})^n \text{ with } p_{11} = 0$$

i.e. $(pt + p'u + p_{oo})^n$

If $p_{11} \neq 0$,polynomials orthogonal to the function generated by $(1+pa +p'b +p_{11}ab)^n$ exist .For example

$$G_{11}(x,y) = n^2(pp'qq'-d^2)G_1(x)G_1(y)-n^2d(p_{10}p_{00}-p_{01}p_{11})G_1(y)$$
$$-n^2d(p_{oo}p_{01}-p_{11}p_{10})G_1(x)-n^3d(pp'qq'-d^2)$$

and

where $d= p_{11}- pp'$

$$S\,S\,G_{11}(x,y)\,w(x,y,d) = 0$$

$$S\,S\,G_{11}(x,y)\,G_1(x)\,w(x,y,d) = 0$$

$$S\,S\,G_{11}(x,y)\,G_1(y)\,w(x,y,d) = 0$$

where $(1 + pa + p'b + p_{11}ab)^n = S\,S\,(1+a)^x(1+b)^y\,w(x,y,d)$

It is evident that these polynomials are more complicated than those discussed in (b) and doubtful whether general expressions can be found for them. We thus consider limiting cases of these polynomials.

Now if p and p' are $= O(1/n)$ and $p_{11} = O(1/n^2)$,putting

$p= m/n$ and $p'=m'/ n$

then $(1 + pa + p'b + p_{11}ab)^n \longrightarrow e^{ma+m'b}$

indicating a frequency function $w(x,y) = \dfrac{e^{-m} m^x}{x!} \dfrac{e^{-m'} m'^y}{y!}$

It is evident that polynomials orthogonal with regard to $w(x,y)$ will be merely products of $K_r(x)$, $K_s(y)$ where $K_r(a)$ is the r^{th} Charlier polynomial in a .

If on the other hand $p=m/n$,$p'=m'/n$ and $p_{11} = O(1/n)$ and $\bar{m} = n(p_{11} - pp')$, then

$$(1 + pa + p'b + p_{11}ab)^n = \left[(1+pa)(1+p'b) + \frac{\bar{m}}{n}ab + O\left(\frac{1}{n^2}\right)\right]^n$$

$$\longrightarrow e^{ma+m'b+\bar{m}ab}$$

which generates

$$w(x,y) = w(x)w(y)\left[1 + \bar{m}K_1(x)K_1(y) + \frac{\bar{m}^2}{2!} K_2(x)K_2(y) + \dots\dots\dots\right]$$

where

$$w(a) = \frac{e^{-m} m^a}{a!}$$

Since in the Gram polynomials in two variables we have assumed $p_{11} = 0$,it is clear that polynomials orthogonal with respect to $w(x,y)$ above are not limiting cases of $G_{rs}(x,y)$.It will be recalled that this is a feature of the one variable case ,for Charlier polynomials can be deduced from Gram's by making n large and p small with $np = m$. With the nucleus function above we shall thus expect to find polynomials in two variables which exhibit certain new features . We shall require the following results for

Charlier polynomials :-

$$K_r(x)K_1(x) = K_{r+1}(x) + \frac{r}{m}K_r(x) + \frac{r}{m}K_{r-1}(x) \qquad \text{........(1)}$$

$$K_r(x)K_2(x) = K_{r+2}(x) + \frac{2r}{m}K_{r+1}(x) + \left(\frac{2r}{m} + \frac{r^{(2)}}{m^2}\right)K_r(x)$$

$$+2\frac{r^{(2)}}{m^2}K_{r-1}(x) + \frac{r^{(2)}}{m^2}K_{r-2}(x) \qquad \text{.......(2)}$$

$$K_r(x)K_3(x) = K_{r+3}(x) + \frac{3r}{m}K_{r+2}(x) + 3r\left(\frac{1}{m} + \frac{(r-1)}{m^2}\right)K_{r+1}(x)$$

$$+\left(\frac{r^{(3)}}{m^3} + 6\frac{r^{(2)}}{m^2}\right)K_r(x) + 3r^{(2)}\left(\frac{1}{m^2} + \frac{(r-2)}{m^3}\right)K_{r-1}(x)$$

$$+3\frac{r^{(3)}}{m^3}K_{r-2}(x) + \frac{r^{(3)}}{m^3}K_{r-3}(x) \qquad \text{..........(3)}$$

$$K_r(x)K_4(x) = K_{r+4}(x) + \frac{4r}{m}K_{r+3}(x) + \left(\frac{4r}{m} + 6\frac{r^{(2)}}{m^2}\right)K_{r+2}(x)$$

$$+\left(12\frac{r^{(2)}}{m^2} + 4\frac{r^{(3)}}{m^3}\right)K_{r+1}(x) + \left(12\frac{r^{(3)}}{m^3} + 6\frac{r^{(2)}}{m^2} + \frac{r^{(4)}}{m^4}\right)K_r(x)$$

$$+\left(12\frac{r^{(3)}}{m^3} + 4\frac{r^{(4)}}{m^4}\right)K_{r-1}(x) + \left(4\frac{r^{(3)}}{m^3} + 6\frac{r^{(4)}}{m^4}\right)K_{r-2}(x)$$

$$+4\frac{r^{(4)}}{m^4}K_{r-3}(x) + \frac{r^{(4)}}{m^4}K_{r-4}(x) \qquad \text{...........(4)}$$

All these results can be deduced from similar expressions for Gram polynomials given earlier ,or by direct substitution. We take $K_s^{(r)}(x,y)$ to represent a Charlier polynomial in x and y orthogonal with respect to

$$w(x,y) = \frac{e^{-m}m^x}{x!}\ \frac{e^{-m'}m'^y}{y!}\left[1 + \bar{m}K_1(x)K_1(y) + \frac{\bar{m}^2}{2!}K_2(x)K_2(y) + \dots\right]$$

e.g. $K_1^{(2)}(x,y) = a_0 + a_1 K_1(x) + a_2 K_2(x) + a_3 K_1(y) + a_4 K_1(x) K_1(y)$

and

$$SS\, K_1^{(2)}(x,y)\, w(x,y) = 0$$

$$SSK_1^{(2)}(x,y)\, w(x,y) K_1(x) = 0$$

$$SS\, K_1^{(2)}(x,y)\, w(x,y)\, K_1(y) = 0$$

$$SS\, K_1^{(2)}(x,y)\, w(x,y)\, K_2(x) = 0$$

giving four equations to determine the four ratios a_1/a_0 , a_2/a_0 , a_3/a_0 , a_4/a_0 . In addition using the expression linear in the a's for $K_1^{(2)}$ we have

$$K_1^{(2)}(x,y) = \begin{vmatrix} 1 & K_1(x) & K_1(y) & K_2(x) & K_1(x)K_1(y) \\ 1 & 0 & 0 & 0 & \frac{\bar{m}}{mm'} \\ 0 & \frac{1}{m} & \frac{\bar{m}}{mm'} & 0 & \frac{\bar{m}}{m'm^2} \\ 0 & \frac{\bar{m}}{mm'} & \frac{1}{m'} & 0 & \frac{\bar{m}}{mm'^2} \\ 0 & 0 & 0 & \frac{2}{m^2} & \frac{2\bar{m}}{m'm^2} \end{vmatrix}$$

using the property

$$\overset{\infty}{\underset{0}{S}}\; K_r^2(x) w(x) = r!\,/m^r$$

and the expression for products of K's on the previous page.

Dividing by a constant ,we find

$$K_1^{(2)}(x,y) = K_1(x)K_1(y) - \frac{\bar{m}}{m'} K_2(x) + \frac{\bar{m}(\bar{m}-m)}{m(mm'-\bar{m}^2)} K_1(y)$$

$$+ \frac{\bar{m}\,(\bar{m}-m')}{m'\,(mm'-\bar{m}^2)} K_1(x) - \frac{\bar{m}}{mm'}$$

Moreover $SS\left[K_1^{(2)}(x,y)\right]^2 w(x,y) = SS\, K_1^{(2)}(x,y)K_1(x)K_1(y)w(x,y)$

since

$$SSK_1^{(2)}(x,y)K_2(x)w(x,y) = SS\, K_1^{(2)}(x,y)K_1(x)w(x,y)$$

$$= SS\, K_1^{(2)}(x,y)K_1(y)w(x,y) = SS\, K_1^{(2)}w(x,y) \quad = 0 \text{ by definition.}$$

Hence we have

$$SS\left[K_1^{(2)}(x,y)\right]^2 w(x,y) = \frac{(mm'-\bar{m}^2)^2 + \bar{m}(\bar{m}-m)(\bar{m}-m')}{m^2\, m'^2(mm'-\bar{m}^2)} \quad (=A \text{ say })$$

Proceeding in this way we find the first few polynomials, namely,

$$K_0^{(1)} = K_1(x) \qquad K_1^{(1)} = K_1(y) - \frac{\bar{m}}{m'}K_1(x)$$

$K_0^{(2)} = K_2(x)$ $\qquad$ $K_1^{(2)}$ is given on the previous page.

$$K_2^{(2)} = \left[1 + \frac{\bar{m}(\bar{m}-m)(\bar{m}-m')}{(mm'-\bar{m}^2)^2}\right]K_2(y) - 2\frac{\bar{m}}{m'}K_1(x)K_1(y)$$

$$+\frac{\bar{m}^2}{m'^2}\left[1 - \frac{\bar{m}(\bar{m}-m)(\bar{m}-m')}{(mm'-\bar{m}^2)^2}\right]K_2(x) - \frac{2\bar{m}^2(\bar{m}-m)}{mm'(mm'-\bar{m}^2)}K_1(y)$$

$$-\frac{2\bar{m}^2(\bar{m}-m')}{m'^2(mm'-m^2)}K_1(x) + \frac{2\bar{m}^2}{mm'^2}$$

Similarly

$$K_1^{(3)} = \left[1 + \frac{\bar{m}(\bar{m}-m)(\bar{m}-m')(mm'+\bar{m}^2)}{(mm'-\bar{m}^2)^3}\right]\left[K_2(x)K_1(y) - \frac{\bar{m}}{m'}K_3(x)\right]$$

$$+\frac{2\bar{m}^3 m'(\bar{m}-m)^2(\bar{m}-m')}{m(mm'-\bar{m}^2)^4}K_2(y) + \frac{2\bar{m}(\bar{m}-m)}{m(mm'-\bar{m}^2)}K_1(x)K_1(y)$$

$$+\frac{2\bar{m}(\bar{m}-m')}{m'(mm'-\bar{m}^2)}\left[1 + \frac{\bar{m}(\bar{m}-m)(\bar{m}^3-mm'^2)}{(mm'-\bar{m}^2)^3}\right]K_2(x) + \frac{2\bar{m}^2(\bar{m}-m)^2}{m^2(mm'-\bar{m}^2)^2}K_1(y)$$

$$-\frac{2\bar{m}}{mm'}\left[1 + \frac{2\bar{m}^3(\bar{m}-m)(\bar{m}-m')}{(mm'-\bar{m}^2)^3}\right]K_1(x) \qquad -\frac{2\bar{m}^2(\bar{m}-m)}{m^2 m'(mm'-\bar{m}^2)}$$

The determinant for $K_1^{(3)}(x,y)$ is the following:-

$$\begin{vmatrix}
1 & K_1(x) & K_1(y) & K_2(x) & K_1(x)K_1(y) & K_2(y) & K_3(x) & K_2(x)K_1(y) \\
1 & 0 & 0 & 0 & \frac{\bar{m}}{mm'} & 0 & 0 & 0 \\
0 & \frac{1}{m} & \frac{\bar{m}}{mm'} & 0 & \frac{\bar{m}}{m^2m'} & 0 & 0 & \frac{2\bar{m}}{m^2m'} \\
0 & \frac{\bar{m}}{mm'} & \frac{1}{m'} & 0 & \frac{\bar{m}}{mm'^2} & 0 & 0 & \frac{2\bar{m}}{m^2m'^2} \\
0 & 0 & 0 & \frac{2}{m^2} & \frac{2\bar{m}}{m^2m'} & \frac{2\bar{m}^2}{m^2m'^2} & 0 & \frac{4\bar{m}}{m^3m'} \\
\frac{\bar{m}}{mm'} & \frac{\bar{m}}{m^2m'} & \frac{\bar{m}}{mm'^2} & \frac{2\bar{m}}{m^2m'} & \left(\frac{2\bar{m}^2}{m^2m'^2}+\frac{\bar{m}}{m^2m'^2}+\frac{1}{mm'}\right) & \frac{2\bar{m}}{mm'^2} & 0 & \left(\frac{4\bar{m}^2}{m^3m'^2}+\frac{2\bar{m}}{m^2m'^2}\right) \\
0 & 0 & 0 & \frac{2\bar{m}^2}{m^2m'^2} & \frac{2\bar{m}}{mm'^2} & \frac{2}{m'^2} & 0 & \frac{4\bar{m}^2}{m^2m'^3} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{m^3} & \frac{6\bar{m}}{m^3m'}
\end{vmatrix}$$

and $K_1^{(3)}(x,y)$ satisfies the following conditions:-

$SS\ K_1^{(3)}\ w(x,y) = SS\ K_1^{(3)}\ w(x,y)K_1(x) = SS\ K_1^{(3)} w(x,y)K_1(y)$

$= SS\ K_1^{(3)} w(x,y)K_2(x) = SS\ K_1^{(3)} w(x,y)K_1(x)K_1(y) = SS\ K_1^{(3)} w(x,y)K_2(y)$

$= SS\ K_1^{(3)} w(x,y)K_3(x) = 0$ and these equations in conjunction with

$$K_1^{(3)}(x,y) = a + bK_1(x) + cK_1(y) + dK_2(x) + eK_1(x)K_1(y) + fK_2(y)$$
$$+ gK_3(x) + hK_2(x)K_1(y)$$

lead to the determinant above.

A general expression for $K^{(r)}_s(x,y)$ has not been found, and it is evident that the determinantal approach is not likely to lead to the desired result. A certain amount of simplification is however introduced by taking $\bar{m}=m'$. We then find

$$K^{(0)} = 1 \qquad K^{(1)}_0 = K_1(x)$$

$$K^{(1)}_1 = K_1(y) - K_1(x)$$

$$K^{(2)}_0 = K_2(x) \qquad K^{(2)}_1 = K_1(x)K_1(y) - K_2(x) - K_1(y)/m - 1/m$$

$$K^{(2)}_2 = K_2(y) - 2K_1(x)K_1(y) + K_2(x) + \frac{2}{m}K_1(y) + \frac{2}{m}$$

$$K^{(3)}_0 = K_3(x)$$

$$K^{(3)}_1 = K_2(x)K_1(y) - K_3(x) - \frac{2}{m}K_1(x)K_1(y) + \frac{2}{m^2}K_1(y) - \frac{2}{m}K_1(x) + \frac{2}{m^2}$$

$$K^{(3)}_2 = K_1(x)K_2(y) - 2K_2(x)K_1(y) + K_3(x) - \frac{2}{m}K_2(y) + \frac{4}{m}K_1(x)K_1(y)$$

$$-\frac{2}{m^2}(m+2)K_1(y) + \frac{4}{m}K_1(x) - \frac{4}{m^2}$$

$$K^{(3)}_3 = K_3(y) + 3K_2(x)K_1(y) - 3K_1(x)K_2(y) - K_3(x) + \frac{6}{m}K_2(y)$$

$$-\frac{6}{m}K_1(x)K_1(y) + \frac{6(m+1)}{m^2}K_1(y) - \frac{6}{m}K_1(x) + \frac{6}{m^2}$$

and

$$SS\left[K^{(1)}_1(x,y)\right]^2 w(x,y) = \frac{m-m'}{mm'} \qquad SS\left[K^{(2)}_1(x,y)\right]^2 w(x,y) = \frac{m-m'}{m^2m'},$$

$$SS\left[K^{(2)}_2(x,y)\right]^2 w(x,y) = 2!\ (m-m')^2/(mm')^2$$

$$SS\left[K^{(3)}_{0}(x,y)\right]^2 w(x,y) = 3!/m^3$$

$$SS\left[K^{(3)}_{1}(x,y)\right]^2 w(x,y) = 2!\ 1!\ (m-m')/m^3m'$$

$$SS\left[K^{(3)}_{2}(x,y)\right]^2 w(x,y) = 2!1!(m-m')^2/m^3m'^2$$

$$SS\left[K^{(3)}_{3}(x,y)\right]^2 w(x,y) = 3!\ (m-m')^3/m^3m'^3$$

Some general results may be stated without proof. They concern the values of $K^{(r)}_{0}$, $K^{(r)}_{1}$, $K^{(r)}_{r-1}$, $K^{(r)}_{r}$. We have

$$K^{(r)}_{0}(x,y) = K_r(x)$$

$$K^{(r)}_{1}(x,y) = K_{r-1}(x)K_1(y) - K_r(x) - \frac{(r-1)}{m}\left[K^{(r-1)}_{1} + K^{(r-1)}_{0} + K^{(r-2)}_{0}\right]$$

$$K^{(r)}_{r-1}(x,y) = K_1(x)K_{r-1}(y) - (r-1)\left[K^{(r)}_{r-2} + \frac{(r-2)}{2}K^{(r)}_{r-3} + \ldots + K^{(r)}_{0}\right] - \frac{(r-1)}{m}\left[K_{r-1}(y) + K_{r-2}(y)\right].$$

$$K^{(r)}_{r}(x,y) = K_r(y) - r\,K^{(r)}_{r-1} - r_{(2)}K^{(r)}_{r-2} - r_{(3)}K^{(r)}_{r-3} - \ldots - K^{(r)}_{0}$$

and finally

$$SS\left[K^{(r)}_{s}(x,y)\right]^2 w(x,y) = (r-s)!\ s!\ (m-m')^s/m^rm'^s \qquad r \geqslant s$$

S U M M A R Y

CHAPTER ONE.

A method of fitting curves to frequency distributions based on Kapteyn's idea of skew frequency is discussed. A method is given to fit the curve

$$y = \frac{F'(x)}{11} e^{-\left[F(x)\right]^2}$$

where $F(x) = A_0 + A_1T_1(x) + A_2T_2(x) + \dots\dots\dots$

and $T_r(x)$ is the $r^{th.}$ orthogonal polynomial of Tchebychef.

This method is applicable to the majority of distributions provided they are not "J" shaped or possess one or two abrupt tails. In the latter case the curve

$$y = \frac{F'(x)}{11} e^{-\left[F(x)\right]^2}$$

where $e^{hF(x)} = A_0 + A_1T_1(x) + A_2T_2(x) + \dots\dots\dots\dots\dots\dots$

can be used successfully. A method of approximation is used to determine the most suitable value of h. Five examples of the first method and four of the second are given.

CHAPTER TWO.

This chapter treats orthogonal polynomials from the point of view of their frequency generating functions.

A determinantal form is found for polynomials orthogonal with respect to w(x) which satisfy

$$(1)\ H_r(x)\ w(x) = \Delta^r\ x^{(r)}\ w(x)$$

$$(2)\ H_r(x)\ w(x) = \Delta^r\ x^{(r)}\ f(x)$$

A general recurrence formula is found for orthogonal polynomials whose frequency generating function is known. Similarly a general relation is established between polynomials and their basic functions. The generating function of $w(x)\ f(\Delta)\ x^{(r)}$ is considered and lastly a new approach to Gram's polynomials set out, from which all the usual properties follow.

CHAPTER THREE.

This deals with orthogonal polynomials in two variables. The continuous case is treated first and polynomials analogous to Hermite's in one variable found. An elegant approach by generating functions leads to the general formulae. In the discrete case, polynomials similar to Gram's and Charlier's are discussed. In the former case general expressions are given which depend on the factorial moment generating function of the product of the polynomial and the nucleus function. No general results could be found in the case of Charlier polynomials in two variables.

In conclusion I would like to express my thanks to Dr. A.C. Aitken for his continued interest and guidance in the subject. The study is a direct outcome of his inspiring

lectures on " Advanced Statistics".

REFERENCES TO LITERATURE.

1. Kapteyn, J.C. "Skew Frequency Curves in Biology and Statistics".

2. A.C.Aitken ,"On the Graduation of Data by the Orthogonal Polynomials of Least Squares". Proc.Roy.Soc.Ed. L 111.

3. W.P.Elderton ,"Frequency Curves and Correlation".

4. S.D.Wicksell, "On the Genetic Theory of Frequency" Arkiv för Maths. Vol. 23 .

5. Udny Yule, "Introduction to the Theory of Statistics".

6. H.L.Rietz, "Frequency distributions obtained by certain transformations of normally distributed variates" Annals of Maths. Vol. 23.

7. S.Bernstein " Sur les Courbes de distribution des probabilités" Math. Zeit. Vol. 24 .

8. C.V.L.Charlier, "Vorlesungen über die Grundzüge der Math. Statistik" 1920.

9. H.E.Greenleaf, "Curve approximation by means of Functions analogous to Hermite polynomials" Annals of Maths. 1932.

10. Aitken and Gonin,"On Fourfold Sampling" .
Proc.Roy.Soc.Ed. Vol.LV Pt.11,1934-35.

11. Aitken,A.C. "On the Orthogonal Polynomials in Frequencies of Type B ". Proc. Roy. Soc. Ed. L 11.

12. Milne-Thomson "Calculus of Finite Differences".

13. A.C.Aitken "Determinants and Matrices". University Mathematical Texts.

14. A.C.Aitken "Mathematical Statistics", University Mathematical Texts.

15. Whittaker and Robinson,"Calculus of Observations".

16. Gonin,H.T. "On the Orthogonal Polynomials in Binomial and Hypergeometric Distributions". A thesis deposited in the Library of the University of Edinburgh.